

On the Robust Dynkin Game ^{*}

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Abstract

We analyze a robust version of the Dynkin game over a set \mathcal{P} of mutually singular probabilities. We first prove that conservative player's lower and upper value coincide (Let us denote the value by V). Such a result connects the robust Dynkin game with second-order doubly reflected backward stochastic differential equations. Also, we show that the value process V is a submartingale under an appropriately defined nonlinear expectation $\underline{\mathcal{E}}$ up to the first time τ_* when V meets the lower payoff process L . If the probability set \mathcal{P} is weakly compact, one can even find an optimal triplet $(\mathbb{P}_*, \tau_*, \gamma_*)$ for the value V_0 .

The mutual singularity of probabilities in \mathcal{P} causes major technical difficulties. To deal with them, we use some new methods including two approximations with respect to the set of stopping times.

Keywords: robust Dynkin game, nonlinear expectation, dynamic programming principle, controls in weak formulation, weak stability under pasting, martingale approach, path-dependent stochastic differential equations with controls, optimal triplet, optimal stopping with random maturity.

1 Introduction

We analyze a continuous-time *robust* Dynkin game with respect to a non-dominated set \mathcal{P} of mutually singular probabilities on the canonical space Ω of continuous paths. In this game, Player 1, who negatively/conservatively thinks that the *Nature* is also against her, will receive the following payment from Player 2 if the two players choose $\tau \in \mathcal{T}$ and $\gamma \in \mathcal{T}$ respectively to quit the game:

$$R(\tau, \gamma) := \int_0^{\tau \wedge \gamma} g_s ds + \mathbf{1}_{\{\tau \leq \gamma\}} L_\tau + \mathbf{1}_{\{\gamma < \tau\}} U_\gamma.$$

Here \mathcal{T} denotes the set of all stopping times with respect to the natural filtration \mathbf{F} of the canonical process B , and the running payoff g , the terminal payoff $L \leq U$ are \mathbf{F} -adapted processes uniformly continuous in sense of (1.6).

As probabilities in \mathcal{P} are mutually singular, one can not define the conditional expectation of the nonlinear expectation $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot]$, and thus Player 1's lower value process \underline{V} and upper value process \overline{V} , in essential extremum sense. Instead, we use shifted processes and regular conditional probability distributions (see Section 1.1 for details) to define

$$\underline{V}_t(\omega) := \sup_{\tau \in \mathcal{T}^t} \inf_{\gamma \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)], \quad \overline{V}_t(\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)], \quad (t, \omega) \in [0, T] \times \Omega.$$

Here \mathcal{T}^t denotes the set of all stopping times with respect to the natural filtration \mathbf{F}^t of the shifted canonical process B^t on the shifted canonical space Ω^t , $\mathcal{P}(t, \omega)$ is a path-dependent probability set which includes all regular conditional probability distributions stemming from \mathcal{P} (see (P2)), and $R^{t, \omega}(\tau, \gamma) := \int_t^{\tau \wedge \gamma} g_s^{t, \omega} ds + \mathbf{1}_{\{\tau \leq \gamma\}} L_\tau^{t, \omega} + \mathbf{1}_{\{\gamma < \tau\}} U_\gamma^{t, \omega}$.

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In Theorem 4.1, we demonstrate that Player 1's lower and upper value processes coincide and thus she has a value process $V_t(\omega) = \underline{V}_t(\omega) = \overline{V}_t(\omega)$, $(t, \omega) \in [0, T] \times \Omega$ in the robust Dynkin game. We also see in Theorem 4.1 that the first time τ_* when V meets L is an optimal stopping time for Player 1, i.e.

$$V_0 = \inf_{\gamma \in \mathcal{T}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[R(\tau_*, \gamma)], \quad (1.1)$$

and that processes $V_t + \int_0^t g_s ds$, $t \in [0, T]$ is a submartingale under the pathwise-defined nonlinear expectation $\underline{\mathcal{E}}_t[\xi](\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}}[\xi^t, \omega]$, $(t, \omega) \in [0, T] \times \Omega$ up to time τ_* .

Since a Dynkin game is actually a coupling of two optimal stopping problems, the *martingale* approach introduced by Snell [55] to solve the optimal stopping problem was later extended to Dynkin games, see e.g. [48, 11, 1, 43, 46]. In the current paper, we will adopt a generalized martingale method with respect to the nonlinear expectations $\underline{\mathcal{E}} = \{\underline{\mathcal{E}}_t\}_{t \in [0, T]}$. The mutual singularity of probabilities in \mathcal{P} gives rise to some major technical hurdles: First, no dominating probability in \mathcal{P} means that we do not have a dominated convergence theorem for the nonlinear expectations $\underline{\mathcal{E}}$. Because of this, one can not follow the classic approach for Dynkin games to obtain the $\underline{\mathcal{E}}$ -martingale property of $V + \int_0^\cdot g_s ds$. Second, we do not have a measurable selection theorem for stopping strategies, which complicates the proof of the dynamic programming principle.

Our martingale approach starts with a dynamic programming principle (DPP) for process \overline{V} . The “subsolution” part of DPP (Proposition 3.1) relies on a “weak stability under pasting” assumption (P3) on the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$, which allows us to construct approximating measures by pasting together local ε -optimal probabilistic models. We show in Section 5 that (P3), along with our other assumptions on the probability class, are satisfied in the case of some path-dependent SDEs with controls, which represents a large class of models on simultaneous drift and volatility uncertainty. We demonstrate that the “supersolution” part of the DPP (Proposition 3.2) by employing a countable dense subset Γ of \mathcal{T}^t to construct a suitable approximation. This dynamic programming result implies the continuity of process \overline{V} (Proposition 3.4), which plays a crucial role in the approximation scheme (to be described in the following paragraph) for proving Theorem 4.1.

The key to Theorem 4.1 is the $\underline{\mathcal{E}}$ -submartingality of process $\{\overline{V}_t + \int_0^t g_s ds\}_{t \in [0, T]}$ up to τ_* . Inspired by Nutz and Zhang [50]'s idea on using stopping times with finitely many values for approximation, we define an approximating sequence of value processes V^n 's to \overline{V} by

$$V_t^n(\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)] \leq \overline{V}_t(\omega), \quad (t, \omega) \in [0, T] \times \Omega,$$

where $\mathcal{T}^t(n)$ collects all \mathcal{T}^t -stopping times taking values in $\{t \vee (i2^{-n}T)\}_{i=0}^{2^n}$. By (P3), Proposition 3.1 still holds for V^n , which leads to that for any $\delta > 0$ and $k \geq n$, the process $\{V_t^n + \int_0^t g_s ds\}_{t \in [0, T]}$ is an $\underline{\mathcal{E}}$ -submartingale over the grid $\{i2^{-k}T\}_{i=0}^{2^k}$ up to the first time $\nu^{n, \delta}$ when V^n meets $L + \delta$ (see (A.14)). Letting $k \rightarrow \infty$, $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we can deduce from $\lim_{n \rightarrow \infty} \uparrow V^n = \overline{V}$ (Proposition 3.3) and the continuity of \overline{V} that the process $\{\overline{V}_t + \int_0^t g_s ds\}_{t \in [0, T]}$ is an $\underline{\mathcal{E}}$ -submartingale up to τ_* . Theorem 4.1 then easily follows. It is worth pointing out that our argument does not require the payoff processes to be bounded.

At the cost of some additional conditions such as the weak compactness of \mathcal{P} and the stronger pasting condition of [56] (all of which are satisfied for *controls of weak formulation*, see Example 6.1), we can apply the main result of [7] to find in Theorem 6.1 a pair $(\mathbb{P}_*, \gamma_*) \in \mathcal{P} \times \mathcal{T}$ such that

$$V_0 = \mathbb{E}_{\mathbb{P}_*}[R(\tau_*, \gamma_*)]. \quad (1.2)$$

Relevant Literature. Since its introduction by [18], Dynkin games have been analyzed in discrete and continuous-time models for decades. Bensoussan and Friedman [24, 8, 9] first analyzed the games in the setting of Markov diffusion processes by means of variational inequalities and free boundary problems. Bayraktar and Sirbu in [4] had a fresh look at this problem using the Stochastic Perron's method (a verification approach without smoothness). For a more general class of reward processes martingale approach was developed under Mokobodzki's condition (see e.g. [48, 10, 11, 1]) and certain regularity assumption on payoff processes (see e.g. [43, 41]).

Cvitanic and Karatzas [16] connected Dynkin games to backward stochastic differential equations (BSDEs) with two reflecting barriers L and U . Along with the growth of the BSDEs theory, Dynkin games have attracted much

attention in the probabilistic framework with Brownian filtration, see e.g. [31, 30, 27, 26, 61, 29, 33, 13, 23, 6]. Among these works, [27, 29, 33, 13, 23, 6] only require “ $L < U$ ” rather than Mokobodski’s condition via a penalization method.

In Mathematical Finance, the theory of Dynkin games can be applied to pricing and hedging game options (or Israeli options) and their derivatives, see [39, 44, 35, 26, 22, 17] and the references in the survey paper [40]. Also, [22, 2] analyzed the sensitivity of the Dynkin game value with respect to changes in the volatility of the underlying. There is plentiful research on Dynkin games in many other areas: for examples, [31, 30, 26, 29, 33] added stochastic *controls* into the Dynkin games to study mixed zero-sum stochastic differential games of control and stopping; [59, 37, 25, 12] and [57, 15] studied some Dynkin games through the associated singular control problems and impulse control problems respectively; [62, 54, 60, 42] considered the Dynkin games in which the players can choose randomized stopping times; and [9, 51, 47, 14, 34, 28, 32] analyzed non-zero sum Dynkin games.

However, there are only a few works on Dynkin games under model uncertainty: Hamadene and Hdhiri [29] and Yin [63] studied the Dynkin games over a set of equivalent probabilities, which represents drift uncertainty (or *Knightian uncertainty*). When the probability set contains mutually singular probabilities (or equivalently, both drift and volatility of the underlying can be “manipulated” against Player 1), Dolinsky [17] derived dual expressions for the superreplication prices of game options in the discrete time, and Matoussi et al. [45] related the Dynkin games under G -expectations (introduced by Peng [52]) to second-order doubly reflected BSDEs.

In this paper we substantially benefit from the martingale techniques developed for robust optimal stopping problems by [38, 3] (which analyzed the problem when \mathcal{P} is dominated), [19] (\mathcal{P} is non-dominated but the Nature and the stopper cooperate) and [50, 5] (in which \mathcal{P} is non-dominated and the Nature and stopper are adversaries.) Especially the results of [7] are crucial for determining a saddle point. (The latter results also recently proved to be useful for defining the viscosity solutions of fully non-linear degenerate path dependent PDEs in [21]).

The rest of the paper is organized as follows: In Section 1.1, we will introduce some notation and preliminary results such as the regular conditional probability distribution. In Section 2, we set-up the stage for our main result by imposing some assumptions on the reward process and the classes of mutually singular probabilities. Then Section 3 derives properties of Player 1’s upper value processes and approximating value processes such as path regularity and dynamic programming principles. They play essential roles in deriving our main result on the robust Dynkin games stated in Section 4. In Section 5, we give an example of path-dependent SDEs with controls that satisfies all our assumptions. In Section 6, we discuss the optimal triplet for Player 1’s value under additional conditions. Section 7 contains proofs of our results while the demonstration of some auxiliary statements with starred labels (in the corresponding equation numbers) in these proofs are deferred to the Appendix. We also include in the appendix a technical lemma necessary for the proof of Theorem 4.1.

1.1 Notation and Preliminaries

Throughout this paper, we fix $d \in \mathbb{N}$. Let $\mathcal{S}_d^{>0}$ stand for all $\mathbb{R}^{d \times d}$ -valued positively definite matrices and denote by $\mathcal{B}(\mathcal{S}_d^{>0})$ the Borel σ -field of $\mathcal{S}_d^{>0}$ under the relative Euclidean topology. We also fix a time horizon $T \in (0, \infty)$ and let $t \in [0, T]$.

We set $\Omega^t := \{\omega \in \mathbb{C}([t, T]; \mathbb{R}^d) : \omega(t) = 0\}$ as the canonical space over period $[t, T]$ and denote its null path by $\mathbf{0}^t := \{\omega(s) = 0, \forall s \in [t, T]\}$. For any $s \in [t, T]$, $\|\omega\|_{t,s} := \sup_{r \in [t,s]} |\omega(r)|$, $\forall \omega \in \Omega^t$ defines a semi-norm on Ω^t . In particular,

$\|\cdot\|_{t,T}$ is the *uniform* norm on Ω^t .

The canonical process B^t of Ω^t is a d -dimensional standard Brownian motion under the Wiener measure \mathbb{P}_0^t of $(\Omega^t, \mathcal{F}_T^t)$. Let $\mathbf{F}^t = \{\mathcal{F}_s^t\}_{s \in [t,T]}$, with $\mathcal{F}_s^t := \sigma(B_r^t; r \in [t, s])$, be the natural filtration of B^t and denote its \mathbb{P}_0^t -augmentation by $\bar{\mathbf{F}}^t = \{\bar{\mathcal{F}}_s^t\}_{s \in [t,T]}$, where $\bar{\mathcal{F}}_s^t := \sigma(\mathcal{F}_s^t \cup \bar{\mathcal{N}}^t)$ and $\bar{\mathcal{N}}^t := \{\mathcal{N} \subset \Omega^t : \mathcal{N} \subset A \text{ for some } A \in \mathcal{F}_T^t \text{ with } \mathbb{P}_0^t(A) = 0\}$. The expectation on $(\Omega^t, \bar{\mathcal{F}}_T^t, \mathbb{P}_0^t)$ will be simply denoted by \mathbb{E}_t . Also, we let \mathcal{P}^t be the \mathbf{F}^t -progressively measurable sigma-field of $[t, T] \times \Omega^t$ and let \mathcal{T}^t (resp. $\bar{\mathcal{T}}^t$) collect all \mathbf{F}^t (resp. $\bar{\mathbf{F}}^t$)-stopping times.

Given $s \in [t, T]$, we set $\mathcal{T}_s^t := \{\tau \in \mathcal{T}^t : \tau(\omega) \geq s, \forall \omega \in \Omega^t\}$, $\bar{\mathcal{T}}_s^t := \{\tau \in \bar{\mathcal{T}}^t : \tau(\omega) \geq s, \forall \omega \in \Omega^t\}$ and define the *truncation* mapping Π_s^t from Ω^t to Ω^s by $(\Pi_s^t(\omega))(r) := \omega(r) - \omega(s)$, $\forall (r, \omega) \in [s, T] \times \Omega^t$. By Lemma A.1 of [5],

$$\tau(\Pi_s^t) \in \mathcal{T}_s^t, \quad \forall \tau \in \mathcal{T}^s. \quad (1.3)$$

For any $\delta > 0$ and $\omega \in \Omega^t$,

$$O_\delta^s(\omega) := \{\omega' \in \Omega^t : \|\omega' - \omega\|_{t,s} < \delta\} \text{ is an } \mathcal{F}_s^t\text{-measurable open set of } \Omega^t, \quad (1.4)$$

and $\overline{O}_\delta^s(\omega) := \{\omega' \in \Omega^t : \|\omega' - \omega\|_{t,s} \leq \delta\}$ is an \mathcal{F}_s^t -measurable closed set of Ω^t (see e.g. (2.1) of [5]). In particular, we will simply denote $O_\delta^T(\omega)$ and $\overline{O}_\delta^T(\omega)$ by $O_\delta(\omega)$ and $\overline{O}_\delta(\omega)$ respectively.

For any $n \in \mathbb{N}$ and $s \in [t, T]$, let $\mathcal{T}^t(n)$ denote all \mathbf{F}^t -stopping times taking values in $\{t_i^n\}_{i=0}^{2^n}$ with

$$t_i^n := t \vee (i2^{-n}T), \quad i = 0, \dots, 2^n, \quad (1.5)$$

and set $\mathcal{T}_s^t(n) := \{\tau \in \mathcal{T}^t(n) : \tau(\omega) \geq s, \forall \omega \in \Omega^t\}$. In particular, we literally set $\mathcal{T}^t(\infty) := \mathcal{T}^t$ and $\mathcal{T}_s^t(\infty) := \mathcal{T}_s^t$.

Let \mathfrak{P}_t collect all probabilities on $(\Omega^t, \mathcal{F}_T^t)$. For any $\mathbb{P} \in \mathfrak{P}_t$, we consider the following spaces about \mathbb{P} :

- 1) For any sub sigma-field \mathcal{G} of \mathcal{F}_T^t , let $L^1(\mathcal{G}, \mathbb{P})$ be the space of all real-valued, \mathcal{G} -measurable random variables ξ with $\|\xi\|_{L^1(\mathcal{G}, \mathbb{P})} := \mathbb{E}_{\mathbb{P}}[|\xi|] < \infty$.
- 2) Let $\mathbb{S}(\mathbf{F}^t, \mathbb{P})$ be the space of all real-valued, \mathbf{F}^t -adapted processes $\{X_s\}_{s \in [t, T]}$ with all continuous paths and satisfying $\mathbb{E}_{\mathbb{P}}[X_*] < \infty$, where $X_* := \|X\|_{t, T} = \sup_{s \in [t, T]} |X_s|$.

We will drop the superscript t from the above notations if it is 0. For example, $(\Omega, \mathcal{F}) = (\Omega^0, \mathcal{F}^0)$.

We say that a process X is bounded by some $C > 0$ if $|X_t(\omega)| \leq C$ for any $(t, \omega) \in [0, T] \times \Omega$. Also, a real-valued process X is said to be uniformly continuous on $[0, T] \times \Omega$ with respect to some modulus of continuity function ρ if

$$|X_{t_1}(\omega_1) - X_{t_2}(\omega_2)| \leq \rho(\mathbf{d}_\infty((t_1, \omega_1), (t_2, \omega_2))), \quad \forall (t_1, \omega_1), (t_2, \omega_2) \in [0, T] \times \Omega, \quad (1.6)$$

where $\mathbf{d}_\infty((t_1, \omega_1), (t_2, \omega_2)) := |t_1 - t_2| + \|\omega_1(\cdot \wedge t_1) - \omega_2(\cdot \wedge t_2)\|_{0, T}$. For any $t \in [0, T]$, taking $t_1 = t_2 = t$ in (1.6) shows that $|X_t(\omega_1) - X_t(\omega_2)| \leq \rho(\|\omega_1 - \omega_2\|_{0, t})$, $\omega_1, \omega_2 \in \Omega$, which implies the \mathcal{F}_t -measurability of X_t . So

X is indeed an \mathbf{F} -adapted process with all continuous paths.

Moreover, let \mathfrak{M} denote all modulus of continuity functions ρ such that for some $C > 0$ and $0 < p_1 \leq p_2$,

$$\rho(x) \leq C(x^{p_1} \vee x^{p_2}), \quad \forall x \in [0, \infty). \quad (1.7)$$

In this paper, we will use the convention $\inf \emptyset := \infty$.

1.2 Shifted Processes and Regular Conditional Probability Distributions

In this subsection, we fix $0 \leq t \leq s \leq T$. The concatenation $\omega \otimes_s \tilde{\omega}$ of an $\omega \in \Omega^t$ and an $\tilde{\omega} \in \Omega^s$ at time s :

$$(\omega \otimes_s \tilde{\omega})(r) := \omega(r) \mathbf{1}_{\{r \in [t, s]\}} + (\omega(s) + \tilde{\omega}(r)) \mathbf{1}_{\{r \in [s, T]\}}, \quad \forall r \in [t, T]$$

defines another path in Ω^t . Set $\omega \otimes_s \emptyset = \emptyset$ and $\omega \otimes_s \tilde{A} := \{\omega \otimes_s \tilde{\omega} : \tilde{\omega} \in \tilde{A}\}$ for any non-empty subset \tilde{A} of Ω^s .

Lemma 1.1. *If $A \in \mathcal{F}_s^t$, then $\omega \otimes_s \Omega^s \subset A$ for any $\omega \in A$.*

For any \mathcal{F}_s^t -measurable random variable η , since $\{\omega' \in \Omega^t : \eta(\omega') = \eta(\omega)\} \in \mathcal{F}_s^t$, Lemma 1.1 implies that

$$\omega \otimes_s \Omega^s \subset \{\omega' \in \Omega^t : \eta(\omega') = \eta(\omega)\} \quad \text{i.e.,} \quad \eta(\omega \otimes_s \tilde{\omega}) = \eta(\omega), \quad \forall \tilde{\omega} \in \Omega^s. \quad (1.8)$$

To wit, the value $\eta(\omega)$ depends only on $\omega|_{[t, s]}$.

Let $\omega \in \Omega^t$. For any $A \subset \Omega^t$ we set $A^{s, \omega} := \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A\}$ as the projection of A on Ω^s along ω . In particular, $\emptyset^{s, \omega} = \emptyset$. Given a random variable ξ on Ω^t , define the *shift* $\xi^{s, \omega}$ of ξ along $\omega|_{[t, s]}$ by $\xi^{s, \omega}(\tilde{\omega}) := \xi(\omega \otimes_s \tilde{\omega})$, $\forall \tilde{\omega} \in \Omega^s$. Correspondingly, for a process $X = \{X_r\}_{r \in [t, T]}$ on Ω^t , its *shifted* process $X^{s, \omega}$ is

$$X^{s, \omega}(r, \tilde{\omega}) := (X_r)^{s, \omega}(\tilde{\omega}) = X_r(\omega \otimes_s \tilde{\omega}), \quad \forall (r, \tilde{\omega}) \in [s, T] \times \Omega^s.$$

Shifted random variables and shifted processes “inherit” the measurability of original ones:

Proposition 1.1. *Let $0 \leq t \leq s \leq T$ and $\omega \in \Omega^t$.*

- (1) *If a real-valued random variable ξ on Ω^t is \mathcal{F}_r^t -measurable for some $r \in [s, T]$, then $\xi^{s,\omega}$ is \mathcal{F}_r^s -measurable.*
- (2) *For any $n \in \mathbb{N} \cup \{\infty\}$ and $\tau \in \mathcal{T}^t(n)$, if $\tau(\omega \otimes_s \Omega^s) \subset [r, T]$ for some $r \in [s, T]$, then $\tau^{s,\omega} \in \mathcal{T}_r^s(n)$.*
- (3) *Given $\tau \in \mathcal{T}^t$, if $\tau(\omega) \leq s$, then $\tau(\omega \otimes_s \Omega^s) \equiv \tau(\omega)$; if $\tau(\omega) \geq s$ (resp. $> s$), then $\tau(\omega \otimes_s \tilde{\omega}) \geq s$ (resp. $> s$), $\forall \tilde{\omega} \in \Omega^s$ and thus $\tau^{s,\omega} \in \mathcal{T}^s$.*
- (4) *If a real-valued process $\{X_r\}_{r \in [t, T]}$ is \mathbf{F}^t -adapted (resp. \mathbf{F}^t -progressively measurable), then $X^{s,\omega}$ is \mathbf{F}^s -adapted (resp. \mathbf{F}^s -progressively measurable).*

Let $\mathbb{P} \in \mathfrak{P}_t$. In light of the *regular conditional probability distributions* (see e.g. [58]), we can follow Section 2.2 of [5] to introduce a family of *shifted probabilities* $\{\mathbb{P}^{s,\omega}\}_{\omega \in \Omega^t} \subset \mathfrak{P}_s$, under which the corresponding shifted random variables and shifted processes inherit the \mathbb{P} integrability of original ones:

Proposition 1.2. (1) *It holds for \mathbb{P}_0^t -a.s. $\omega \in \Omega^t$ that $(\mathbb{P}_0^t)^{s,\omega} = \mathbb{P}_0^s$.*

(2) *If $\xi \in L^1(\mathcal{F}_T^t, \mathbb{P})$ for some $\mathbb{P} \in \mathfrak{P}_t$, then it holds for \mathbb{P} -a.s. $\omega \in \Omega^t$ that $\xi^{s,\omega} \in L^1(\mathcal{F}_T^s, \mathbb{P}^{s,\omega})$ and*

$$\mathbb{E}_{\mathbb{P}^{s,\omega}}[\xi^{s,\omega}] = \mathbb{E}_{\mathbb{P}}[\xi | \mathcal{F}_s^t](\omega) \in \mathbb{R}. \quad (1.9)$$

(3) *If $X \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$ for some $\mathbb{P} \in \mathfrak{P}_t$, then it holds for \mathbb{P} -a.s. $\omega \in \Omega^t$ that $X^{s,\omega} \in \mathbb{S}(\mathbf{F}^s, \mathbb{P}^{s,\omega})$.*

As a consequence of (1.9), a shifted \mathbb{P}_0^t -null set also has zero measure.

Lemma 1.2. *For any $\mathcal{N} \in \overline{\mathcal{N}}^t$, it holds for \mathbb{P}_0^t -a.s. $\omega \in \Omega^t$ that $\mathcal{N}^{s,\omega} \in \overline{\mathcal{N}}^s$.*

This subsection was presented in [5] with more details and proofs. In the next three sections, we will gradually provide the technical set-up and preparation for our main results (Theorem 4.1 and Theorem 6.1) on the robust Dynkin game.

2 Weak Stability under Pasting

To study the robust Dynkin game, we need some regularity conditions on the payoff processes.

Standing assumptions on payoff processes (g, L, U).

(A) g, L and U are three real-valued processes that are uniformly continuous on $[0, T] \times \Omega$ with respect to the same modulus of continuity function ρ_0 and satisfy $L_t(\omega) \leq U_t(\omega)$, $\forall (t, \omega) \in [0, T] \times \Omega$.

For any $(t, \omega) \in [0, T] \times \Omega$ and $s, s' \in [t, T]$, we technically define $R(t, s, s', \omega) := \int_t^{s \wedge s'} g_r(\omega) dr + \mathbf{1}_{\{s \leq s'\}} L_s(\omega) + \mathbf{1}_{\{s' < s\}} U_{s'}(\omega)$. By (1.6),

$$\begin{aligned} |R(t, s, s', \omega_1) - R(t, s, s', \omega_2)| &\leq \int_t^{s \wedge s'} |g_r(\omega_1) - g_r(\omega_2)| dr + \mathbf{1}_{\{s \leq s'\}} |L_s(\omega_1) - L_s(\omega_2)| + \mathbf{1}_{\{s' < s\}} |U_{s'}(\omega_1) - U_{s'}(\omega_2)| \\ &\leq (1 + s \wedge s' - t) \rho_0(\|\omega_1 - \omega_2\|_{0, s \wedge s'}), \quad \forall \omega_1, \omega_2 \in \Omega. \end{aligned} \quad (2.1)$$

Let the robust Dynkin game start from time $t \in [0, T]$ when the history has been evolving along path $\omega|_{[0, t]}$ for some $\omega \in \Omega$. Player 1 and 2 make their own choices on the exiting time of the game. If Player 1 selects $\tau \in \mathcal{T}^t$ and Player 2 selects $\gamma \in \mathcal{T}^t$, the game ceases at $\tau \wedge \gamma$. Then Player 1 will receive from her opponent an accumulated reward $\int_t^{\tau \wedge \gamma} g_s^{t,\omega} ds$ and a terminal payoff $L_\tau^{t,\omega}$ (resp. $U_\gamma^{t,\omega}$) if $\tau \leq \gamma$ (resp. $\gamma < \tau$). Here negative $\int_t^{\tau \wedge \gamma} g_s^{t,\omega} ds$, $L_\tau^{t,\omega}$ or $U_\gamma^{t,\omega}$ means a payment from Player 1 to Player 2. So Player 1's total wealth at time $\tau \wedge \gamma$ is

$$R^{t,\omega}(\tau, \gamma) := \int_t^{\tau \wedge \gamma} g_s^{t,\omega} ds + \mathbf{1}_{\{\tau \leq \gamma\}} L_\tau^{t,\omega} + \mathbf{1}_{\{\gamma < \tau\}} U_\gamma^{t,\omega} = \int_t^{\tau \wedge \gamma} g_s^{t,\omega} ds + \mathbf{1}_{\{\tau \leq \gamma\}} L_{\tau \wedge \gamma}^{t,\omega} + \mathbf{1}_{\{\gamma < \tau\}} U_{\tau \wedge \gamma}^{t,\omega}.$$

Since Proposition 1.1 (4) shows that $g^{t,\omega}$, $L^{t,\omega}$ and $U^{t,\omega}$ are \mathbf{F}^t -adapted processes with all continuous paths,

$$R^{t,\omega}(\tau, \gamma) \in \mathcal{F}_{\tau \wedge \gamma}^t, \quad \forall \tau, \gamma \in \mathcal{T}^t. \quad (2.2)$$

Also, it is clear that

$$(R^{t,\omega}(\tau, \gamma))(\tilde{\omega}) = R(t, \tau(\tilde{\omega}), \gamma(\tilde{\omega}), \omega \otimes_t \tilde{\omega}), \quad \forall \tilde{\omega} \in \Omega^t. \quad (2.3)$$

Next, we define $\Psi_t := (-L_t) \vee U_t \vee 0$, $t \in [0, T]$. By (1.6), one can deduce that

$$|\Psi_t(\omega_1) - \Psi_t(\omega_2)| \leq \rho_0(\|\omega_1 - \omega_2\|_{0,t}), \quad \forall t \in [0, T], \quad \forall \omega_1, \omega_2 \in \Omega; \quad (2.4)$$

(For the reader's convenience we provided a proof in Section 7.1.)

It is clear that

$$|R^{t,\omega}(\tau, \gamma)| \leq \int_t^{\tau \wedge \gamma} |g_s^{t,\omega}| ds + \Psi_{\tau \wedge \gamma}^{t,\omega}, \quad \forall (t, \omega) \in [0, T] \times \Omega, \quad \forall \tau, \gamma \in \mathcal{T}^t. \quad (2.5)$$

The following result shows that the integrability of shifted payoff processes is independent of the given path history.

Lemma 2.1. *Assume (A). For any $t \in [0, T]$ and $\mathbb{P} \in \mathfrak{P}_t$, if $\Psi^{t,\omega} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$ and $\mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,\omega}| ds < \infty$ for some $\omega \in \Omega$, then $\Psi^{t,\omega'} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$ and $\mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,\omega'}| ds < \infty$ for all $\omega' \in \Omega$.*

We will concentrate on those probabilities \mathbb{P} in \mathfrak{P}_t under which shifted payoff processes are integrable:

Assumption 2.1. *For any $t \in [0, T]$, $\widehat{\mathfrak{P}}_t := \left\{ \mathbb{P} \in \mathfrak{P}_t : \Psi^{t,0} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P}) \text{ and } \mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,0}| ds < \infty \right\}$ is not empty.*

Remark 2.1. (1) If $\Psi \in \mathbb{S}(\mathbf{F}, \mathbb{P}_0)$ and $\mathbb{E}_{\mathbb{P}_0} \int_0^T |g_s| ds < \infty$, then $\mathbb{P}_0^t \in \widehat{\mathfrak{P}}_t$ for any $t \in [0, T]$. (2) As we will show in Proposition 5.1, when the modulus of continuity ρ_0 in (A) has polynomial growth, the laws of solutions to the controlled SDEs (5.1) over period $[t, T]$ belong to $\widehat{\mathfrak{P}}_t$.

Under (A) and Assumption 2.1, one can deduce from Lemma 2.1 that for any $t \in [0, T]$ and $\mathbb{P} \in \widehat{\mathfrak{P}}_t$,

$$\Psi^{t,\omega} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P}) \quad \text{and} \quad \mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,\omega}| ds < \infty, \quad \forall \omega \in \Omega. \quad (2.6)$$

Next, we need the probability class to be adapted and weakly stable under pasting in the following sense:

Standing assumptions on the probability class.

(P1) For any $t \in [0, T]$, we consider a family $\{\mathcal{P}(t, \omega)\}_{\omega \in \Omega}$ of subsets of $\widehat{\mathfrak{P}}_t$ such that

$$\mathcal{P}(t, \omega_1) = \mathcal{P}(t, \omega_2) \quad \text{if} \quad \omega_1|_{[0,t]} = \omega_2|_{[0,t]}. \quad (2.7)$$

Assume further that the probability class $\{\mathcal{P}(t, \omega)\}_{(t,\omega) \in [0,T] \times \Omega}$ satisfy the following two conditions for some modulus of continuity function $\widehat{\rho}_0$: for any $0 \leq t < s \leq T$, $\omega \in \Omega$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$:

(P2) There exists an extension $(\Omega^t, \mathcal{F}', \mathbb{P}')$ of $(\Omega^t, \mathcal{F}_T^t, \mathbb{P})$ (i.e. $\mathcal{F}_T^t \subset \mathcal{F}'$ and $\mathbb{P}'|_{\mathcal{F}_T^t} = \mathbb{P}$) and $\Omega' \in \mathcal{F}'$ with $\mathbb{P}'(\Omega') = 1$ such that $\mathbb{P}^{s,\tilde{\omega}}$ belongs to $\mathcal{P}(s, \omega \otimes_t \tilde{\omega})$ for any $\tilde{\omega} \in \Omega'$.

(P3) (*weak stability under pasting*) For any $\delta \in \mathbb{Q}_+$ and $\lambda \in \mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^\lambda$ be a \mathcal{F}_s^t -partition of Ω^t such that for $j=1, \dots, \lambda$, $\mathcal{A}_j \subset O_{\delta_j}^s(\tilde{\omega}_j)$ for some $\delta_j \in ((0, \delta] \cap \mathbb{Q}) \cup \{\delta\}$ and $\tilde{\omega}_j \in \Omega^t$. Then for any $\mathbb{P}_j \in \mathcal{P}(s, \omega \otimes_t \tilde{\omega}_j)$, $j=1, \dots, \lambda$, there exists a $\widehat{\mathbb{P}} \in \mathcal{P}(t, \omega)$ such that

- (i) $\widehat{\mathbb{P}}(A \cap \mathcal{A}_0) = \mathbb{P}(A \cap \mathcal{A}_0)$, $\forall A \in \mathcal{F}_T^t$;
- (ii) For any $j=1, \dots, \lambda$ and $A \in \mathcal{F}_s^t$, $\widehat{\mathbb{P}}(A \cap \mathcal{A}_j) = \mathbb{P}_j(A \cap \mathcal{A}_j)$;
- (iii) For any $n \in \mathbb{N} \cup \{\infty\}$ and $\wp \in \mathcal{T}^s$, there exist $\wp_j^n \in \mathcal{T}_s^t$, $j=1, \dots, \lambda$ such that for any $A \in \mathcal{F}_s^t$ and $\tau \in \mathcal{T}_s^t(n)$

$$\sum_{j=1}^\lambda \mathbb{E}_{\widehat{\mathbb{P}}} [\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \wp_j^n)] \leq \sum_{j=1}^\lambda \mathbb{E}_{\mathbb{P}_j} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] + \int_t^s g_r^{t,\omega}(\tilde{\omega}) dr \right) \right] + \widehat{\rho}_0(\delta). \quad (2.8)$$

Remark 2.2. (1) By (2.7), one can regard $\mathcal{P}(t, \omega)$ as a path-dependent subset of $\widehat{\mathfrak{P}}_t$. In particular, $\mathcal{P} := \mathcal{P}(0, 0) = \mathcal{P}(0, \omega)$, $\forall \omega \in \Omega$.

(2) Both sides of (2.8) are finite as we will show in Section 7. In particular, the expectations on the right-hand-side are well-defined since the mapping $\tilde{\omega} \rightarrow \sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \tilde{\omega}}(\varsigma, \wp)]$ is continuous under norm $\|\cdot\|_{t,T}$ for any $n \in \mathbb{N} \cup \{\infty\}$,

$\widehat{\mathbb{P}} \in \widehat{\mathfrak{P}}_s$ and $\wp \in \mathcal{T}^s$.

(3) Analogous to (P2) assumed in [5], the condition (P3) can be regarded as a weak form of stability under pasting since it is implied by the “stability under finite pasting” (see e.g. (4.18) of [56]): for any $0 \leq t < s \leq T$, $\omega \in \Omega$, $\mathbb{P} \in \mathcal{P}(t, \omega)$, $\delta \in \mathbb{Q}_+$ and $\lambda \in \mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^\lambda$ be a \mathcal{F}_s^t -partition of Ω^t such that for $j = 1, \dots, \lambda$, $\mathcal{A}_j \subset O_{\delta_j}^s(\tilde{\omega}_j)$ for some $\delta_j \in ((0, \delta] \cap \mathbb{Q}) \cup \{\delta\}$ and $\tilde{\omega}_j \in \Omega^t$. Then for any $\mathbb{P}_j \in \mathcal{P}(s, \omega \otimes_t \tilde{\omega}_j)$, $j = 1, \dots, \lambda$, the probability defined by

$$\widehat{\mathbb{P}}(A) = \mathbb{P}(A \cap \mathcal{A}_0) + \sum_{j=1}^\lambda \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in \mathcal{A}_j\}} \mathbb{P}_j(A^{s, \tilde{\omega}}) \right], \quad \forall A \in \mathcal{F}_T^t \quad (2.9)$$

is in $\mathcal{P}(t, \omega)$.

As pointed out in Remark 3.6 of [49] (see also Remark 3.4 of [5]), (2.9) is not suitable for the example of path-dependent SDEs with controls (see Section 5). Thus we assume the weak pasting condition (P3), which turns out to be sufficient for our approximation scheme in proving the main results.

3 The Dynamic Programming Principle

Consider the robust Dynkin game with payoff processes (g, L, U) and over the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ as described in Section 2. If Player 1 conservatively thinks that *Nature* is also against her, then for any $(t, \omega) \in [0, T] \times \Omega$,

$$\underline{V}_t(\omega) := \sup_{\tau \in \mathcal{T}^t} \inf_{\gamma \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)] \quad \text{and} \quad \overline{V}_t(\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)]$$

define the *lower* value and *upper* value of Player 1 at time t given the historical path $\omega|_{[0, t]}$.

As we will see in Theorem 4.1 that \underline{V} coincides with \overline{V} as Player 1's value process V , whose sum with $\int_0^\cdot g_s ds$ is an \mathcal{E} -submartingale up to the first time τ_* when V meets L . For this purpose, we derive in this section some basic properties of \overline{V} and its approximating values including dynamic programming principles. Let (A), (P1)–(P3) and Assumption 2.1 hold throughout the section.

For any $(t, \omega) \in [0, T] \times \Omega$, following [50]'s idea, we technically define approximating value processes of \overline{V} by

$$V_t^n(\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)] \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \gamma)] = \overline{V}_t(\omega), \quad \forall n \in \mathbb{N}, \quad (3.1)$$

and set in particular $V_t^\infty(\omega) := \overline{V}_t(\omega)$.

Let $n \in \mathbb{N} \cup \{\infty\}$. It is clear that

$$V^n(T, \omega) = \inf_{\mathbb{P} \in \mathcal{P}(T, \omega)} \inf_{\gamma \in \mathcal{T}^T} \sup_{\tau \in \mathcal{T}^T(n)} \mathbb{E}_{\mathbb{P}}[R^{T, \omega}(\tau, \gamma)] = \inf_{\mathbb{P} \in \mathcal{P}(T, \omega)} \mathbb{E}_{\mathbb{P}}[R^{T, \omega}(T, T)] = L_T(\omega), \quad \forall \omega \in \Omega. \quad (3.2)$$

And we can show that

$$-\Psi_t(\omega) \leq L_t(\omega) \leq V_t^n(\omega) \leq U_t(\omega) \leq \Psi_t(\omega), \quad \forall (t, \omega) \in [0, T] \times \Omega. \quad (3.3)$$

For the reader's convenience we provide a proof in Section 7.1.

We need the following assumption on V^n 's to discuss the dynamic programming principles they satisfy.

Assumption 3.1. *There exists a modulus of continuity function $\rho_1 \geq \rho_0$ such that for any $n \in \mathbb{N} \cup \{\infty\}$*

$$|V_t^n(\omega_1) - V_t^n(\omega_2)| \leq \rho_1(\|\omega_1 - \omega_2\|_{0, t}), \quad \forall t \in [0, T], \quad \forall \omega_1, \omega_2 \in \Omega. \quad (3.4)$$

Remark 3.1. *If $\mathcal{P}(t, \omega)$ does not depend on ω for all $t \in [0, T]$, then Assumption 3.1 holds automatically.*

Remark 3.2. *Assumption 3.1 implies that V^n is \mathbf{F} -adapted for any $n \in \mathbb{N} \cup \{\infty\}$.*

We first present the sub-solution side of dynamic programming principle for V^n 's:

Proposition 3.1. *For any $n \in \mathbb{N} \cup \{\infty\}$, $0 \leq t \leq s \leq T$ and $\omega \in \Omega$,*

$$V_t^n(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t, \omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left((V_s^n)^{t, \omega} + \int_t^s g_r^{t, \omega} dr \right) \right]. \quad (3.5)$$

Conversely, we only need to show the super-solution side of dynamic programming principle for $V^\infty = \bar{V}$.

Proposition 3.2. *For any $0 \leq t \leq s \leq T$ and $\omega \in \Omega$,*

$$\bar{V}_t(\omega) \geq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t, \omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left(\bar{V}_s^{t, \omega} + \int_t^s g_r^{t, \omega} dr \right) \right].$$

As a consequence of Propositions 3.1 and 3.2, the upper value process \bar{V} of Player 1 satisfies a true dynamic programming principle.

We rely on another condition to further show the convergence of V^n to \bar{V} and their path regularities in the next two propositions.

Assumption 3.2. *For any $\alpha > 0$, there exists a modulus of continuity function ρ_α such that for any $t \in [0, T]$*

$$\sup_{\omega \in O_\alpha^t(\mathbf{0})} \sup_{\mathbb{P} \in \mathcal{P}(t, \omega)} \sup_{\zeta \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} \left[\rho_1 \left(\delta + \sup_{r \in [\zeta, (\zeta + \delta) \wedge T]} |B_r^t - B_\zeta^t| \right) \right] \leq \rho_\alpha(\delta), \quad \forall \delta \in (0, T]. \quad (3.6)$$

Proposition 3.3. *Let $n \in \mathbb{N}$, $t \in [0, T]$ and $\alpha > 0$. It holds for any $\omega \in O_\alpha^t(\mathbf{0})$ that*

$$\bar{V}_t(\omega) \leq V_t^n(\omega) + \rho_\alpha(2^{-n}) + 2^{-n}(|g_t(\omega)| + \rho_\alpha(T-t)). \quad (3.7)$$

Proposition 3.4. (1) *For any $n \in \mathbb{N} \cup \{\infty\}$, all paths of process V^n are both left-upper-semicontinuous and right-lower-semicontinuous. In particular, the process \bar{V} has all continuous paths.*

(2) *For any $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$, $\bar{V}^{t, \omega} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$.*

4 Main Result

In this section, we state our first main result on robust Dynkin games. Let (A), (P1)–(P3) and Assumptions 2.1, 3.1, 3.2 hold throughout the section.

Given $t \in [0, T]$, set $\mathcal{L}_t := \{\text{random variable } \xi \text{ on } \Omega : \xi^{t, \omega} \in L^1(\mathcal{F}_T^t, \mathbb{P}), \forall \omega \in \Omega, \forall \mathbb{P} \in \mathcal{P}(t, \omega)\}$. Clearly, \mathcal{L}_t is closed under linear combination: i.e. for any $\xi_1, \xi_2 \in \mathcal{L}_t$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \xi_1 + \alpha_2 \xi_2 \in \mathcal{L}_t$. Then we define on \mathcal{L}_t a nonlinear expectation:

$$\underline{\mathcal{E}}_t[\xi](\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}}[\xi^{t, \omega}], \quad \forall \omega \in \Omega, \forall \xi \in \mathcal{L}_t.$$

For any $n \in \mathbb{N} \cup \{\infty\}$ and $\tau \in \mathcal{T}$,

$$\text{both } V_\tau^n \text{ and } \int_0^\tau g_r dr \text{ belong to } \mathcal{L}_t. \quad (4.1)$$

(We demonstrate this claim in Section 7.3.)

Similar to the classic Dynkin game, we will show that \bar{V} coincides with \underline{V} as the value process V of Player 1 in the robust Dynkin game and that V plus $\int_0^t g_s ds$ is a submartingale with respect to the nonlinear expectation $\underline{\mathcal{E}}$.

Theorem 4.1. *Let (A), (P1)–(P3) and Assumptions 2.1, 3.1, 3.2 hold.*

(1) *For any $(t, \omega) \in [0, T] \times \Omega$,*

$$V_t(\omega) := \underline{V}_t(\omega) = \bar{V}_t(\omega) \quad (4.2)$$

in the robust Dynkin game starting from time t given the historical path $\omega|_{[0, t]}$. Moreover,

$$V_t(\omega) = \inf_{\gamma \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_{(t, \omega)}^*, \gamma)], \quad \text{where } \tau_{(t, \omega)}^* := \inf \{s \in [t, T] : V_s^{t, \omega} = L_s^{t, \omega}\} \in \mathcal{T}^t. \quad (4.3)$$

(2) *The \mathbf{F} -adapted process with all continuous paths $\Upsilon_t := V_t + \int_0^t g_r dr$, $t \in [0, T]$ is an $\underline{\mathcal{E}}$ -submartingale up to time $\tau_* := \tau_{(0, \mathbf{0})}^* = \inf \{t \in [0, T] : V_t = L_t\} \in \mathcal{T}$ in sense that for any $\zeta \in \mathcal{T}$*

$$\Upsilon_{\tau_* \wedge \zeta \wedge t}(\omega) \leq \underline{\mathcal{E}}_t[\Upsilon_{\tau_* \wedge \zeta}](\omega), \quad \forall (t, \omega) \in [0, T] \times \Omega. \quad (4.4)$$

5 Examples: Controlled Path-dependent SDEs

In this section, we provide an example of the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ in case of path-dependent stochastic differential equations with controls.

Let $\kappa > 0$ and let $b: [0, T] \times \Omega \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^d$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d \times d}) / \mathcal{B}(\mathbb{R}^d)$ -measurable function such that

$$|b(t, \omega, u) - b(t, \omega', u)| \leq \kappa \|\omega - \omega'\|_{0, t} \quad \text{and} \quad |b(t, \mathbf{0}, u)| \leq \kappa(1 + |u|), \quad \forall \omega, \omega' \in \Omega, (t, u) \in [0, T] \times \mathbb{R}^{d \times d}.$$

Fix $t \in [0, T]$. We let \mathcal{U}_t collect all $\mathcal{S}_d^>0$ -valued, \mathbf{F}^t -progressively measurable processes $\{\mu_s\}_{s \in [t, T]}$ such that $|\mu_s| \leq \kappa$, $ds \times d\mathbb{P}_0^t$ -a.s. Let $\omega \in \Omega$, $b^{t, \omega}(r, \tilde{\omega}, u) := b(r, \omega \otimes_t \tilde{\omega}, u)$, $(r, \tilde{\omega}, u) \in [t, T] \times \Omega^t \times \mathbb{R}^{d \times d}$ is clearly a $\mathcal{P}^t \otimes \mathcal{B}(\mathbb{R}^{d \times d}) / \mathcal{B}(\mathbb{R}^d)$ -measurable function that satisfies

$$|b^{t, \omega}(r, \tilde{\omega}, u) - b^{t, \omega}(r, \tilde{\omega}', u)| \leq \kappa \|\tilde{\omega} - \tilde{\omega}'\|_{t, r} \quad \text{and} \quad |b^{t, \omega}(r, \mathbf{0}^t, u)| \leq \kappa(1 + \|\omega\|_{0, t} + |u|), \quad \forall \tilde{\omega}, \tilde{\omega}' \in \Omega^t, (r, u) \in [t, T] \times \mathbb{R}^{d \times d}.$$

Given $\mu \in \mathcal{U}_t$, a slight extension of Theorem V.12.1 of [53] shows that the following SDE on the probability space $(\Omega^t, \mathcal{F}_T^t, \mathbb{P}_0^t)$:

$$X_s = \int_t^s b^{t, \omega}(r, X_r, \mu_r) dr + \int_t^s \mu_r dB_r^t, \quad s \in [t, T], \quad (5.1)$$

admits a unique solution $X^{t, \omega, \mu}$, which is an $\bar{\mathbf{F}}^t$ -adapted continuous process satisfying $E_t[(X_*^{t, \omega, \mu})^p] < \infty$ for any $p \geq 1$ (or see the complete ArXiv version of [5] for its proof).

Note that the SDE (5.1) depends on $\omega|_{[0, t]}$ via the generator $b^{t, \omega}$. Without loss of generality, we assume that all paths of $X^{t, \omega, \mu}$ are continuous and starting from 0. (Otherwise, by setting $\mathcal{N} := \{\omega \in \Omega^t : X_t^{t, \omega, \mu}(\omega) \neq \mathbf{0} \text{ or the path } X^{t, \omega, \mu}(\omega) \text{ is not continuous}\} \in \bar{\mathcal{N}}^t$, one can take $\tilde{X}_s^{t, \omega, \mu} := \mathbf{1}_{\mathcal{N}^c} X_s^{t, \omega, \mu}$, $s \in [t, T]$. It is an $\bar{\mathbf{F}}^t$ -adapted process that satisfies (5.1) and whose paths are all continuous and starting from 0.)

Applying the Burkholder-Davis-Gundy inequality, Gronwall's inequality and using the Lipschitz continuity of b in ω -variable, one can easily derive the following estimates for $X^{t, \omega, \mu}$: for any $p \geq 1$

$$\mathbb{E}_t \left[\sup_{r \in [t, s]} |X_r^{t, \omega, \mu} - X_r^{t, \omega', \mu}|^p \right] \leq C_p \|\omega - \omega'\|_{0, t}^p (s - t)^p, \quad \forall \omega' \in \Omega, \quad \forall s \in [t, T], \quad (5.2)$$

$$\text{and} \quad \mathbb{E}_t \left[\sup_{r \in [\zeta, (\zeta + \delta) \wedge T]} |X_r^{t, \omega, \mu} - X_\zeta^{t, \omega, \mu}|^p \right] \leq \varphi_p(\|\omega\|_{0, t}) \delta^{p/2}, \quad \text{for any } \bar{\mathbf{F}}^t\text{-stopping time } \zeta \text{ and } \delta > 0, \quad (5.3)$$

where C_p is a constant depending on p, κ, T and $\varphi_p: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function depending on p, κ, T (see the complete ArXiv version of [5] for the proofs of (5.2) and (5.3)).

For any $s \in [t, T]$, we see from [5] that $\mathcal{F}_s^t \subset \mathcal{G}_s^{X^{t, \omega, \mu}} := \{A \subset \Omega^t : (X^{t, \omega, \mu})^{-1}(A) \in \bar{\mathcal{F}}_s^t\}$, i.e.,

$$(X^{t, \omega, \mu})^{-1}(A) \in \bar{\mathcal{F}}_s^t, \quad \forall A \in \mathcal{F}_s^t. \quad (5.4)$$

Namely, $X^{t, \omega, \mu}$ is $\bar{\mathcal{F}}_s^t / \mathcal{F}_s^t$ -measurable as a mapping from Ω^t to Ω^t . Define the law of $X^{t, \omega, \mu}$ under \mathbb{P}_0^t by

$$\mathbf{p}^{t, \omega, \mu}(A) := \mathbb{P}_0^t \circ (X^{t, \omega, \mu})^{-1}(A), \quad \forall A \in \mathcal{G}_T^{X^{t, \omega, \mu}},$$

and denote by $\mathbb{P}^{t, \omega, \mu}$ the restriction of $\mathbf{p}^{t, \omega, \mu}$ on $(\Omega^t, \mathcal{F}_T^t)$.

Now, let us set $\mathcal{P}(t, \omega) := \{\mathbb{P}^{t, \omega, \mu} : \mu \in \mathcal{U}_t\} \subset \mathfrak{P}_t$.

Proposition 5.1. *Let ϱ_0 be a modulus of continuity function such that for some $\varpi \geq 1$, $\varrho_0(\delta) \leq \kappa(1 + \delta^\varpi)$, $\forall \delta > 0$. Assume that g, L, U satisfy (A) with respect to ϱ_0 and that $\int_0^T |g_t(\mathbf{0})| dt < \infty$. Then for any $(t, \omega) \in [0, T] \times \Omega$, we have $\mathcal{P}(t, \omega) \subset \hat{\mathfrak{P}}_t$. And the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ satisfies (P1)–(P3), Assumption 3.1–3.2.*

Remark 5.1. (1) When $b \equiv 0$, Proposition 5.1 and the result (4.2) verify Assumption 5.7 of [45] (particularly for $t = 0$). Then we know from Theorem 5.8 therein that in case of controlled path-dependent SDEs with null drift, Player 1's value V is closely related to the solution of a second-order doubly reflected backward stochastic differential equation.

(2) Similar to [5], the reason we consider the law of $X^{t,\omega,\mu}$ under \mathbb{P}_0^t over $\mathcal{G}_T^{X^{t,\omega,\mu}}$ (the largest σ -field to induce \mathbb{P}_0^t under the mapping $X^{t,\omega,\mu}$) rather than \mathcal{F}_T^t lies in the fact that the proof of Proposition 5.1 relies heavily on the inverse mapping $W^{t,\omega,\mu}$ of $X^{t,\omega,\mu}$. According to the proofs of Proposition 6.2 and 6.3 in [5], since $W^{t,\omega,\mu}$ is an \mathbf{F}^t -progressively measurable processes that has only $\mathbf{p}^{t,\omega,\mu}$ -a.s. continuous paths, it holds for $\mathbf{p}^{t,\omega,\mu}$ -a.s. $\tilde{\omega} \in \Omega^t$ that the shifted probability $(\mathbb{P}^{t,\omega,\mu})^{s,\tilde{\omega}}$ is the law of the solution to the shifted SDE (and thus $(\mathbb{P}^{t,\omega,\mu})^{s,\tilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \tilde{\omega})$). This explains why our assumption (P2) needs an extension $(\Omega^t, \mathcal{F}', \mathbb{P}')$ of the probability space $(\Omega^t, \mathcal{F}_T^t, \mathbb{P})$.

6 The Optimal Triplet

In this section, we identify an optimal triplet for Player 1's value in the robust Dynkin game under the following additional conditions on the payoff processes and the probability class.

(A') Let $g \equiv 0$ and let L, U be two real-valued processes bounded by some $M_0 > 0$ such that they are uniformly continuous on $[0, T] \times \Omega$ with respect to the same $\rho_0 \in \mathfrak{M}$, that $L_t(\omega) \leq U_t(\omega)$, $\forall (t, \omega) \in [0, T] \times \Omega$, and that $L_T(\omega) = U_T(\omega)$, $\forall \omega \in \Omega$.

Also, let a family $\{\mathcal{P}_t\}_{t \in [0, T]}$ of subsets \mathcal{P}_t of $\hat{\mathfrak{P}}_t = \mathfrak{P}_t$, $t \in [0, T]$ satisfy:

(H1) $\mathcal{P} := \mathcal{P}_0$ is a weakly compact subset of \mathfrak{P}_0 .

(H2) For any $\rho \in \mathfrak{M}$, there exists another $\bar{\rho}$ of \mathfrak{M} such that

$$\sup_{(\mathbb{P}, \zeta) \in \mathcal{P}_t \times \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} \left[\rho \left(\delta + \sup_{r \in [\zeta, (\zeta + \delta) \wedge T]} |B_r^t - B_{\zeta}^t| \right) \right] \leq \bar{\rho}(\delta), \quad \forall t \in [0, T], \quad \forall \delta \in (0, \infty).$$

In particular, we require $\bar{\rho}_0$ to satisfy (1.7) with some $\bar{C} > 0$ and $1 < \bar{p}_1 \leq \bar{p}_2$.

(H3) For any $0 \leq t < s \leq T$, $\omega \in \Omega$ and $\mathbb{P} \in \mathcal{P}_t$, there exists an extension $(\Omega^t, \mathcal{F}', \mathbb{P}')$ of $(\Omega^t, \mathcal{F}_T^t, \mathbb{P})$ (i.e. $\mathcal{F}_T^t \subset \mathcal{F}'$ and $\mathbb{P}'|_{\mathcal{F}_T^t} = \mathbb{P}$) and $\Omega' \in \mathcal{F}'$ with $\mathbb{P}'(\Omega') = 1$ such that $\mathbb{P}^{s, \tilde{\omega}}$ belongs to \mathcal{P}_s for any $\tilde{\omega} \in \Omega'$.

(H4) Moreover, let the finite stability under pasting stated in Remark 2.2 (3) hold.

The next example shows that *controls of weak formulation* (i.e. \mathcal{P} contains all semimartingale measures under which B has uniformly bounded drift and diffusion coefficients) satisfies (H1)–(H4).

Example 6.1. Given $\ell > 0$, let $\{\mathcal{P}_t^\ell\}_{t \in [0, T]}$ be the family of semimartingale measures considered in [20] such that \mathcal{P}_t^ℓ collects all continuous semimartingale measures on $(\Omega^t, \mathcal{F}_T^t)$ whose drift and diffusion characteristics are bounded by ℓ and $\sqrt{2\ell}$ respectively. According to Lemma 2.3 therein, $\{\mathcal{P}_t^\ell\}_{t \in [0, T]}$ satisfies (H1), (H3) and (H4). Also, one can deduce from the Burkholder-Davis-Gundy inequality that $\{\mathcal{P}_t^\ell\}_{t \in [0, T]}$ satisfies (H2), see the proof of [7, Example 3.3] for details.

Remark 2.2 (3) and a revisit of Remark 3.1's proof show that the path-independent probability class $\{\mathcal{P}_t\}_{t \in [0, T]}$ satisfies (P1)–(P3) and Assumption 3.1 with $\rho_1 = \rho_0$, while Assumption 3.2 is clearly implied by (H2) with $\rho_\alpha \equiv \bar{\rho}_0$, $\forall \alpha > 0$. So Theorem 4.1 still holds for the robust Dynkin game over $\{\mathcal{P}_t\}_{t \in [0, T]}$. In addition, (H1) enables us to apply the result of [7] to solve (1.2).

Theorem 6.1. Under Assumptions (A') and (H1)–(H4), there exists a pair $(\mathbb{P}_*, \gamma_*) \in \mathcal{P} \times \mathcal{T}$ such that $V_0 = \mathbb{E}_{\mathbb{P}_*}[R(\tau_*, \gamma_*)]$.

Remark 6.1. Theorem 4.1 (1) and Theorem 6.1 imply that

$$V_0 = \mathbb{E}_{\mathbb{P}_*}[R(\tau_*, \gamma_*)] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[R(\tau_*, \gamma_*)] \geq \inf_{\gamma \in \mathcal{T}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[R(\tau_*, \gamma)] = V_0,$$

which shows that $V_0 = \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[R(\tau_*, \gamma_*)] = \underline{\mathcal{E}}_0[R(\tau_*, \gamma_*)]$. Hence, we see that the pair (τ_*, γ_*) is robust with respect to $\mathbb{P} \in \mathcal{P}$, or (τ_*, γ_*) is a saddle point of the Dynkin game under the nonlinear expectation $\underline{\mathcal{E}}_0$.

7 Proofs

7.1 Proofs of technical results in Sections 1.1, 2 and 3

Proof of Proposition 1.1 (2): Let $n \in \mathbb{N}$ and $\tau \in \mathcal{T}^t(n)$. Assume that $\tau(\omega \otimes_s \Omega^s) \subset [r, T]$ for some $r \in [s, T]$. For any $i = 0, \dots, 2^n$ such that $t_i^n = t \vee (i2^{-n}T) \geq r$, since $r \geq s \geq t$, one has $\tilde{r} := t \vee (i2^{-n}T) = (t \vee (i2^{-n}T)) \vee s = s \vee (i2^{-n}T)$.

Setting $A := \{\omega' \in \Omega^t : \tau(\omega') \leq \tilde{r}\} \in \mathcal{F}_{\tilde{r}}^t$, we can deduce from Lemma 2.2 of [5] that

$$\{\tilde{\omega} \in \Omega^s : \tau^{s,\omega}(\tilde{\omega}) \leq \tilde{r}\} = \{\tilde{\omega} \in \Omega^s : \tau(\omega \otimes_s \tilde{\omega}) \leq \tilde{r}\} = \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A\} = A^{s,\omega} \in \mathcal{F}_{\tilde{r}}^s.$$

So $\tau^{s,\omega}$ is an \mathbf{F}^s -stopping time valued in $\{t \vee (i2^{-n}T) \in [r, T] : i=0, \dots, 2^n\} \subset \{s \vee (i2^{-n}T) \in [r, T] : i=0, \dots, 2^n\}$, i.e. $\tau^{s,\omega} \in \mathcal{T}_r^s(n)$.

For the case of $n=\infty$, see Corollary 2.1 of [5]. \square

Proof of (2.4): Let $t \in [0, T]$ and $\omega_1, \omega_2 \in \Omega$. We see from (1.6) that

$$\begin{aligned} -L_t(\omega_1) &\leq -L_t(\omega_2) + |L_t(\omega_1) - L_t(\omega_2)| \leq \Psi_t(\omega_2) + \rho_0(\|\omega_1 - \omega_2\|_{0,t}), \\ \text{and } U_t(\omega_1) &\leq U_t(\omega_2) + |U_t(\omega_1) - U_t(\omega_2)| \leq \Psi_t(\omega_2) + \rho_0(\|\omega_1 - \omega_2\|_{0,t}). \end{aligned}$$

It follows that $\Psi_t(\omega_1) = (-L_t(\omega_1)) \vee U_t(\omega_1) \vee 0 \leq \Psi_t(\omega_2) + \rho_0(\|\omega_1 - \omega_2\|_{0,t})$. Then exchanging the roles of ω_1 and ω_2 proves (2.4). \square

Proof of Lemma 2.1: Let $t \in [0, T]$ and $\mathbb{P} \in \mathfrak{P}_t$. Suppose that $\Psi^{t,\omega} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$ and $\mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,\omega}| ds < \infty$ for some $\omega \in \Omega$. Let $\omega' \in \Omega$. For any $(s, \tilde{\omega}) \in [t, T] \times \Omega^t$, (1.6) implies that

$$|g_s^{t,\omega'}(\tilde{\omega}) - g_s^{t,\omega}(\tilde{\omega})| = |g_s(\omega' \otimes_t \tilde{\omega}) - g_s(\omega \otimes_t \tilde{\omega})| \leq \rho_0(\|\omega' \otimes_t \tilde{\omega} - \omega \otimes_t \tilde{\omega}\|_{0,s}) = \rho_0(\|\omega' - \omega\|_{0,t}), \quad (7.1)$$

so $\mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,\omega'}| ds \leq \mathbb{E}_{\mathbb{P}} \int_t^T |g_s^{t,\omega}| ds + (T-t)\rho_0(\|\omega' - \omega\|_{0,t}) < \infty$.

Proposition 1.1 (4) shows that both $L^{t,\omega'}$ and $U^{t,\omega'}$ are \mathbf{F}^t -adapted processes with all continuous paths, so is the process $\Psi_s^{t,\omega'} = (-L_s^{t,\omega'}) \vee U_s^{t,\omega'} \vee 0$, $s \in [t, T]$. Similar to (7.1), we see from (2.4) that

$$|\Psi_s^{t,\omega'}(\tilde{\omega}) - \Psi_s^{t,\omega}(\tilde{\omega})| \leq \rho_0(\|\omega' - \omega\|_{0,t}), \quad \forall (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

It follows that $\mathbb{E}_{\mathbb{P}}[\Psi_*^{t,\omega'}] = \mathbb{E}_{\mathbb{P}}\left[\sup_{s \in [t, T]} |\Psi_s^{t,\omega'}|\right] \leq \mathbb{E}_{\mathbb{P}}\left[\sup_{s \in [t, T]} |\Psi_s^{t,\omega}|\right] + \rho_0(\|\omega' - \omega\|_{0,t}) = \mathbb{E}_{\mathbb{P}}[\Psi_*^{t,\omega}] + \rho_0(\|\omega' - \omega\|_{0,t}) < \infty$.

Therefore, $\Psi^{t,\omega'} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$. \square

Proof of Remark 2.1 (1): Let $t \in [0, T]$. Proposition 1.2 implies that for \mathbb{P}_0 -a.s. $\omega \in \Omega$, $\Psi^{t,\omega} \in \mathbb{S}(\mathbf{F}^t, (\mathbb{P}_0)^{t,\omega}) = \mathbb{S}(\mathbf{F}^t, \mathbb{P}_0^t)$ and

$$\mathbb{E}_{\mathbb{P}_0^t} \int_t^T |g_s^{t,\omega}| ds = \mathbb{E}_{(\mathbb{P}_0)^{t,\omega}} \left[\left(\int_t^T |g_s| ds \right)^{t,\omega} \right] \leq \mathbb{E}_{(\mathbb{P}_0)^{t,\omega}} \left[\left(\int_0^T |g_s| ds \right)^{t,\omega} \right] = \mathbb{E}_{\mathbb{P}_0} \left[\int_0^T |g_s| ds \middle| \mathcal{F}_t \right] (\omega) < \infty.$$

It then follows from Lemma 2.1 that $\Psi^{t,0} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P}_0^t)$ and $\mathbb{E}_{\mathbb{P}_0^t} \int_t^T |g_s^{t,0}| ds < \infty$. Hence, $\mathbb{P}_0^t \in \hat{\mathfrak{P}}_t$. \square

Proof of Remark 2.2: 2) Fix $t \in [0, T]$ and let $\omega_1, \omega_2 \in \Omega$, $\tau, \gamma \in \mathcal{T}^t$. By (2.3) and (2.1),

$$\begin{aligned} |(R^{t,\omega_1}(\tau, \gamma))(\tilde{\omega}) - (R^{t,\omega_2}(\tau, \gamma))(\tilde{\omega})| &= |R(t, \tau(\tilde{\omega}), \gamma(\tilde{\omega}), \omega_1 \otimes_t \tilde{\omega}) - R(t, \tau(\tilde{\omega}), \gamma(\tilde{\omega}), \omega_2 \otimes_t \tilde{\omega})| \\ &\leq (1+T)\rho_0(\|\omega_1 \otimes_t \tilde{\omega} - \omega_2 \otimes_t \tilde{\omega}\|_{0,T}) = (1+T)\rho_0(\|\omega_1 - \omega_2\|_{0,t}), \quad \forall \tilde{\omega} \in \Omega^t. \end{aligned} \quad (7.2)$$

Now, let $\omega \in \Omega$, $s \in [t, T]$, $n \in \mathbb{N} \cup \{\infty\}$, $\tilde{\mathbb{P}} \in \hat{\mathfrak{P}}_s$ and $\wp \in \mathcal{T}^s$. Given $\tilde{\omega}_1, \tilde{\omega}_2 \in \Omega^t$ and $\varsigma \in \mathcal{T}^s(n)$, similar to (7.2),

$$|R^{s,\omega \otimes_t \tilde{\omega}_1}(\varsigma, \wp) - R^{s,\omega \otimes_t \tilde{\omega}_2}(\varsigma, \wp)| \leq (1+T)\rho_0(\|\omega \otimes_t \tilde{\omega}_1 - \omega \otimes_t \tilde{\omega}_2\|_{0,s}) = (1+T)\rho_0(\|\tilde{\omega}_1 - \tilde{\omega}_2\|_{t,s}). \quad (7.3)$$

It follows that $\mathbb{E}_{\tilde{\mathbb{P}}} [R^{s,\omega \otimes_t \tilde{\omega}_1}(\varsigma, \wp)] \leq \mathbb{E}_{\tilde{\mathbb{P}}} [R^{s,\omega \otimes_t \tilde{\omega}_2}(\varsigma, \wp)] + (1+T)\rho_0(\|\tilde{\omega}_1 - \tilde{\omega}_2\|_{t,s})$. Taking supremum over $\varsigma \in \mathcal{T}^s(n)$ yields that $\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\tilde{\mathbb{P}}} [R^{s,\omega \otimes_t \tilde{\omega}_1}(\varsigma, \wp)] \leq \sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\tilde{\mathbb{P}}} [R^{s,\omega \otimes_t \tilde{\omega}_2}(\varsigma, \wp)] + (1+T)\rho_0(\|\tilde{\omega}_1 - \tilde{\omega}_2\|_{t,T})$. Exchanging the roles of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ shows that the mapping $\tilde{\omega} \rightarrow \sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\tilde{\mathbb{P}}} [R^{s,\omega \otimes_t \tilde{\omega}}(\varsigma, \wp)]$ is continuous under norm $\|\cdot\|_{t,T}$ and thus \mathcal{F}_T^t -measurable.

Next, let us show that both sides of (2.8) are finite: Let $A \in \mathcal{F}_s^t$, $\tau \in \mathcal{T}_s^t(n)$ and $j=1, \dots, \lambda$. By (2.5) and (2.6),

$$|\mathbb{E}_{\tilde{\mathbb{P}}} [\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \wp_j^n)]| \leq \mathbb{E}_{\tilde{\mathbb{P}}} [|R^{t,\omega}(\tau, \wp_j^n)|] \leq \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_t^{\tau \wedge \wp_j^n} |g_s^{t,\omega}| ds + \Psi_{\tau \wedge \wp_j^n}^{t,\omega} \right] \leq \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_t^T |g_s^{t,\omega}| ds + \Psi_*^{t,\omega} \right] < \infty.$$

On the other hand, given $\tilde{\omega} \in A \cap \mathcal{A}_j$ and $\varsigma \in \mathcal{T}^s(n)$, taking $(\tilde{\omega}_1, \tilde{\omega}_2) = (\tilde{\omega}, \tilde{\omega}_j)$ in (7.3), we can deduce from (2.5) and (2.6) again that

$$\left| \mathbb{E}_{\mathbb{P}_j} [R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] \right| \leq \mathbb{E}_{\mathbb{P}_j} \left[\left| R^{s, \omega \otimes_t \tilde{\omega}_j}(\varsigma, \wp) \right| \right] + (1+T)\rho_0(\|\tilde{\omega} - \tilde{\omega}_j\|_{t,s}) \leq \mathbb{E}_{\mathbb{P}_j} \left[\int_s^T |g_r^{s, \omega \otimes_t \tilde{\omega}_j}| dr + \Psi_*^{s, \omega \otimes_t \tilde{\omega}_j} \right] + (1+T)\rho_0(\delta) := \alpha_j < \infty.$$

It then follows that

$$\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr \right) \right] \leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{A \cap \mathcal{A}_j} \int_t^T |g_r^{t, \omega}| dr \right] + \alpha_j \mathbb{P}(A \cap \mathcal{A}_j) < \infty,$$

as well as that

$$\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr \right) \right] \geq -\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{A \cap \mathcal{A}_j} \int_t^T |g_r^{t, \omega}| dr \right] - \alpha_j \mathbb{P}(A \cap \mathcal{A}_j) > -\infty.$$

Summing both up over $j \in \{1, \dots, \lambda\}$ shows that the right-hand-side of (2.8) is finite.

3) The proof of Remark 3.3 (2) in [5] has shown that the probability $\widehat{\mathbb{P}}$ defined in (2.9) satisfies (P3) (i) and (ii): $\widehat{\mathbb{P}}(A \cap \mathcal{A}_0) = \mathbb{P}(A \cap \mathcal{A}_0)$, $\forall A \in \mathcal{F}_T^t$, and $\widehat{\mathbb{P}}(A \cap \mathcal{A}_j) = \mathbb{P}(A \cap \mathcal{A}_j)$, $\forall j = 1, \dots, \lambda$, $\forall A \in \mathcal{F}_s^t$. To see $\widehat{\mathbb{P}}$ satisfying (2.8), let us fix $n \in \mathbb{N} \cup \{\infty\}$ and $\wp \in \mathcal{T}^s$. We set $\wp_j^n := \wp(\Pi_s^t)$, $j = 1, \dots, \lambda$, which are of \mathcal{T}_s^t by (1.3).

Let $A \in \mathcal{F}_s^t$ and $\tau \in \mathcal{T}_s^t(n)$. Given $\tilde{\omega} \in \Omega^t$, Proposition 1.1 (2) shows that $\tau^{s, \tilde{\omega}} \in \mathcal{T}^s(n)$. Since the \mathbf{F} -adaptedness of g and (1.8) imply that

$$g_r(\omega \otimes_t \Omega^t) = g_r(\omega), \quad \forall r \in [0, t] \quad \text{and} \quad g_r((\omega \otimes_t \tilde{\omega}) \otimes_s \Omega^s) = g_r(\omega \otimes_t \tilde{\omega}), \quad \forall r \in [0, s], \quad (7.4)$$

we see from (2.3) that for any $\hat{\omega} \in \Omega^s$

$$\begin{aligned} (R^{t, \omega}(\tau, \wp_j^n))^{s, \tilde{\omega}}(\hat{\omega}) &= (R^{t, \omega}(\tau, \wp_j^n))(\tilde{\omega} \otimes_s \hat{\omega}) = R(t, \tau(\tilde{\omega} \otimes_s \hat{\omega}), \wp(\Pi_s^t(\tilde{\omega} \otimes_s \hat{\omega})), \omega \otimes_t (\tilde{\omega} \otimes_s \hat{\omega})) \\ &= R(s, \tau^{s, \tilde{\omega}}(\hat{\omega}), \wp(\hat{\omega}), (\omega \otimes_t \tilde{\omega}) \otimes_s \hat{\omega}) + \int_t^s g_r((\omega \otimes_t \tilde{\omega}) \otimes_s \hat{\omega}) dr = (R^{s, \omega \otimes_t \tilde{\omega}}(\tau^{s, \tilde{\omega}}, \wp))(\hat{\omega}) + \int_t^s g_r(\omega \otimes_t \tilde{\omega}) dr. \end{aligned} \quad (7.5)$$

By Lemma 1.1, $(A \cap \mathcal{A}_j)^{s, \tilde{\omega}} = \Omega^s$ (resp. $= \emptyset$) if $\tilde{\omega} \in A \cap \mathcal{A}_j$ (resp. $\notin A \cap \mathcal{A}_j$). Then (7.5) leads to that

$$\begin{aligned} \mathbb{E}_{\widehat{\mathbb{P}}} [\mathbf{1}_{A \cap \mathcal{A}_j} R^{t, \omega}(\tau, \wp_j^n)] &= \sum_{j'=1}^{\lambda} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in \mathcal{A}_{j'}\}} \mathbb{E}_{\mathbb{P}_{j'}} \left[(\mathbf{1}_{A \cap \mathcal{A}_j} R^{t, \omega}(\tau, \wp_j^n))^{s, \tilde{\omega}} \right] \right] \\ &= \sum_{j'=1}^{\lambda} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \mathbf{1}_{\{\tilde{\omega} \in \mathcal{A}_{j'}\}} \mathbb{E}_{\mathbb{P}_{j'}} \left[(R^{t, \omega}(\tau, \wp_j^n))^{s, \tilde{\omega}} \right] \right] = \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\mathbb{E}_{\mathbb{P}_j} \left[R^{s, \omega \otimes_t \tilde{\omega}}(\tau^{s, \tilde{\omega}}, \wp) \right] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr \right) \right] \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr \right) \right]. \end{aligned}$$

Taking summation over $j \in \{1, \dots, \lambda\}$ yields (2.8). \square

Proof of (3.3): Let $(t, \omega) \in [0, T] \times \Omega$. Since the \mathcal{F}_t -measurability of L_t , U_t and (1.8) show that

$$L_t^{t, \omega}(\tilde{\omega}) = L_t(\omega \otimes_t \tilde{\omega}) = L_t(\omega) \quad \text{and} \quad U_t^{t, \omega}(\tilde{\omega}) = U_t(\omega \otimes_t \tilde{\omega}) = U_t(\omega), \quad \forall \tilde{\omega} \in \Omega^t. \quad (7.6)$$

it holds for any $\tau \in \mathcal{T}^t(n)$ that $R^{t, \omega}(\tau, t) = \mathbf{1}_{\{\tau=t\}} L_{\tau}^{t, \omega} + \mathbf{1}_{\{\tau < t\}} U_{\tau}^{t, \omega} = \mathbf{1}_{\{\tau=t\}} L_t^{t, \omega} + \mathbf{1}_{\{\tau < t\}} U_t^{t, \omega} \leq U_t^{t, \omega} = U_t(\omega)$. So

$$V_t^n(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau, t)] \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [U_t(\omega)] = U_t(\omega) \leq \Psi_t(\omega).$$

On the other hand, since $t \in \mathcal{T}^t(n)$ and since $R^{t, \omega}(t, \gamma) = \mathbf{1}_{\{t \leq \gamma\}} L_t^{t, \omega} + \mathbf{1}_{\{t < \gamma\}} U_{\gamma}^{t, \omega} = L_t^{t, \omega} = L_t(\omega)$ for any $\gamma \in \mathcal{T}^t$,

$$V_t^n(\omega) \geq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \inf_{\gamma \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(t, \gamma)] = \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [L_t(\omega)] = L_t(\omega) \geq -\Psi_t(\omega). \quad \square$$

Proof of Remark 3.1: Fix $n \in \mathbb{N} \cup \{\infty\}$. Let $t \in [0, T]$, $\omega_1, \omega_2 \in \Omega$, $\mathbb{P} \in \mathcal{P}_t$ and $\tau, \gamma \in \mathcal{T}^t$. By (7.2), $\mathbb{E}_{\mathbb{P}} [R^{t, \omega_1}(\tau, \gamma)] \leq \mathbb{E}_{\mathbb{P}} [R^{t, \omega_2}(\tau, \gamma)] + (1+T)\rho_0(\|\omega_1 - \omega_2\|_{0,t})$. Taking supremum over $\tau \in \mathcal{T}^t(n)$, taking infimum over $\gamma \in \mathcal{T}^t$ and then taking infimum over $\mathbb{P} \in \mathcal{P}_t$ yield that $V_t^n(\omega_1) \leq V_t^n(\omega_2) + (1+T)\rho_0(\|\omega_1 - \omega_2\|_{0,t})$. Exchanging the roles of ω_1 and ω_2 , we obtain (3.4) with $\rho_1 = (1+T)\rho_0$ for each $n \in \mathbb{N} \cup \{\infty\}$. \square

7.2 Proofs of the Dynamic Programming Principles

Proof of Proposition 3.1: Fix $n \in \mathbb{N} \cup \{\infty\}$, $0 \leq t \leq s \leq T$ and $\omega \in \Omega$.

1) When $t = s$, since V^n is \mathbf{F} -adapted by Remark 3.2, an analogy to (7.6) shows that $(V^n)_t^{t,\omega}(\tilde{\omega}) = V^n(t, \omega \otimes_t \tilde{\omega}) = V_t^n(\omega)$, $\forall \tilde{\omega} \in \Omega^t$. Then

$$\inf_{\mathbb{P} \in \mathcal{P}(t,\omega)} \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < t\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq t\}} (V_t^n)^{t,\omega} \right] = \inf_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}_{\mathbb{P}} [V_t^n(\omega)] = V_t^n(\omega).$$

2) To demonstrate (3.5) for case $t < s$, we shall paste the local approximating \mathbb{P} -minimizers of $(V_s^n)^{t,\omega}$ according to (P3) and then make some estimations.

2a) Under norm $\|\cdot\|_{t,T}$, since Ω^t is a separable complete metric space, there exists a countable dense subset $\{\hat{\omega}_j^t\}_{j \in \mathbb{N}}$ of Ω^t . Fix $\varepsilon > 0$ and let $\delta \in \mathbb{Q}_+$ satisfy $\rho_1(\delta) \vee \rho_0(\delta) \vee ((1+T)\rho_0(\delta)) < \varepsilon/5$. Let $j \in \mathbb{N}$. By (1.4), $\mathcal{A}_j := O_\delta^s(\hat{\omega}_j^t) \setminus (\bigcup_{j' < j} O_\delta^s(\hat{\omega}_{j'}^t)) \in \mathcal{F}_s^t$. We can find a $\mathbb{P}_j \in \mathcal{P}(s, \omega \otimes_t \hat{\omega}_j^t)$ and a $\gamma_j \in \mathcal{T}^s$ such that

$$V_s^n(\omega \otimes_t \hat{\omega}_j^t) \geq \inf_{\gamma \in \mathcal{T}^s} \sup_{\tau \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \hat{\omega}_j^t}(\tau, \gamma)] - \frac{1}{5}\varepsilon \geq \sup_{\tau \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \hat{\omega}_j^t}(\tau, \gamma_j)] - \frac{2}{5}\varepsilon. \quad (7.7)$$

Given $\tilde{\omega} \in O_\delta^s(\hat{\omega}_j^t)$, an analogy to (7.3) shows that for any $\tau \in \mathcal{T}^s(n)$

$$|R^{s,\omega \otimes_t \tilde{\omega}}(\tau, \gamma_j) - R^{s,\omega \otimes_t \hat{\omega}_j^t}(\tau, \gamma_j)| \leq (1+T)\rho_0(\|\tilde{\omega} - \hat{\omega}_j^t\|_{t,s}) \leq (1+T)\rho_0(\delta) \leq \frac{1}{5}\varepsilon,$$

so $\mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \tilde{\omega}}(\tau, \gamma_j)] \leq \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \hat{\omega}_j^t}(\tau, \gamma_j)] + \varepsilon/5$. Taking supremum over $\tau \in \mathcal{T}^s(n)$, we see from (7.7) and (3.4) that

$$\begin{aligned} \sup_{\tau \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \tilde{\omega}}(\tau, \gamma_j)] &\leq \sup_{\tau \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \hat{\omega}_j^t}(\tau, \gamma_j)] + \frac{1}{5}\varepsilon \leq V_s^n(\omega \otimes_t \hat{\omega}_j^t) + \frac{3}{5}\varepsilon \leq V_s^n(\omega \otimes_t \tilde{\omega}) + \rho_1(\|\omega \otimes_t \tilde{\omega} - \omega \otimes_t \hat{\omega}_j^t\|_{0,s}) + \frac{3}{5}\varepsilon \\ &= (V_s^n)^{t,\omega}(\tilde{\omega}) + \rho_1(\|\tilde{\omega} - \hat{\omega}_j^t\|_{t,s}) + \frac{3}{5}\varepsilon \leq (V_s^n)^{t,\omega}(\tilde{\omega}) + \rho_1(\delta) + \frac{3}{5}\varepsilon \leq (V_s^n)^{t,\omega}(\tilde{\omega}) + \frac{4}{5}\varepsilon. \end{aligned} \quad (7.8)$$

Next, fix $\mathbb{P} \in \mathcal{P}(t, \omega)$, $\lambda \in \mathbb{N}$ and let $\hat{\mathbb{P}}_\lambda$ be the probability of $\mathcal{P}(t, \omega)$ in (P3) for $\{(\mathcal{A}_j, \delta_j, \tilde{\omega}_j, \mathbb{P}_j)\}_{j=1}^\lambda = \{(\mathcal{A}_j, \delta, \hat{\omega}_j^t, \mathbb{P}_j)\}_{j=1}^\lambda$ and $\mathcal{A}_0 := \left(\bigcup_{j=1}^\lambda \mathcal{A}_j\right)^c \in \mathcal{F}_s^t$. Then we have

$$\mathbb{E}_{\hat{\mathbb{P}}_\lambda}[\xi] = \mathbb{E}_{\mathbb{P}}[\xi], \quad \forall \xi \in L^1(\mathcal{F}_s^t, \hat{\mathbb{P}}_\lambda) \cap L^1(\mathcal{F}_s^t, \mathbb{P}) \quad \text{and} \quad \mathbb{E}_{\hat{\mathbb{P}}_\lambda}[\mathbf{1}_{\mathcal{A}_0}\xi] = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\mathcal{A}_0}\xi], \quad \forall \xi \in L^1(\mathcal{F}_T^t, \hat{\mathbb{P}}_\lambda) \cap L^1(\mathcal{F}_T^t, \mathbb{P}). \quad (7.9)$$

Also, in light of (2.8) and (7.8), there exist $\wp_j^n \in \mathcal{T}_s^t$, $j=1, \dots, \lambda$, such that for any $A \in \mathcal{F}_s^t$ and $\tau \in \mathcal{T}_s^t(n)$

$$\begin{aligned} \sum_{j=1}^\lambda \mathbb{E}_{\hat{\mathbb{P}}_\lambda} [\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \wp_j^n)] &\leq \sum_{j=1}^\lambda \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}_j} [R^{s,\omega \otimes_t \tilde{\omega}}(\varsigma, \gamma_j)] + \int_t^s g_r^{t,\omega}(\tilde{\omega}) dr \right) \right] + \hat{\rho}_0(\delta) \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{A \cap \mathcal{A}_0^c} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + \varepsilon. \end{aligned} \quad (7.10)$$

2b) Now, let $\gamma \in \mathcal{T}^t$ and $\tau \in \mathcal{T}^t(n)$. Applying (7.10) with $A = \{\tau \wedge \gamma \geq s\} \in \mathcal{F}_s^t$, one can show that

$$\sum_{j=1}^\lambda \mathbb{E}_{\hat{\mathbb{P}}_\lambda} [\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_j} R^{t,\omega}(\tau, \wp_j^n)] \leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_0^c} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + \varepsilon. \quad (7.11^*)$$

We glue γ with $\{\wp_j^n\}_{j=1}^\lambda$ to form a new \mathbf{F}^t -stopping time

$$\hat{\gamma}_\lambda := \mathbf{1}_{\{\gamma < s\}} \gamma + \mathbf{1}_{\{\gamma \geq s\}} \left(\mathbf{1}_{\mathcal{A}_0} \gamma + \sum_{j=1}^\lambda \mathbf{1}_{\mathcal{A}_j} \wp_j^n \right). \quad (7.12^*)$$

Since $\hat{\gamma}_\lambda \geq s > \tau$ on $\{\gamma \geq s\} \cap \{\tau < s\}$, (2.2) shows that

$$\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \hat{\gamma}_\lambda) = \mathbf{1}_{\{\gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\gamma \geq s\} \cap \{\tau < s\}} \left(\int_t^\tau g_s^{t,\omega} ds + L_\tau^{t,\omega} \right) = \mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) \in \mathcal{F}_s^t.$$

Then one can deduce from (7.9), (7.11), (2.5) and (3.3) that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_\lambda} [R^{t,\omega}(\tau, \hat{\gamma}_\lambda)] &= \mathbb{E}_{\mathbb{P}_\lambda} [(\mathbf{1}_{\{\tau \wedge \gamma < s\}} + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \cap \mathcal{A}_0) R^{t,\omega}(\tau, \gamma)] + \sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}_\lambda} [\mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \cap \mathcal{A}_j R^{t,\omega}(\tau, \wp_j^n)] \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \cap \mathcal{A}_0 \left(R^{t,\omega}(\tau, \gamma) - (V_s^n)^{t,\omega} - \int_t^s g_r^{t,\omega} dr \right) \right] + \varepsilon \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) + \mathbf{1}_{\mathcal{A}_0} \left(2 \int_t^T |g_r^{t,\omega}| dr + 2\Psi_*^{t,\omega} \right) \right] + \varepsilon. \end{aligned}$$

Taking supremum over $\tau \in \mathcal{T}^t(n)$ yields that

$$V_t^n(\omega) \leq \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + 2\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\mathcal{A}_0} \left(\int_t^T |g_r^{t,\omega}| dr + \Psi_*^{t,\omega} \right) \right] + \varepsilon.$$

Then taking infimum over $\gamma \in \mathcal{T}^t$ on the right-hand-side, we obtain

$$V_t^n(\omega) \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + 2\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\left(\bigcup_{j=1}^{\lambda} \mathcal{A}_j\right)^c} \left(\int_t^T |g_r^{t,\omega}| dr + \Psi_*^{t,\omega} \right) \right] + \varepsilon.$$

Since $\bigcup_{j \in \mathbb{N}} \mathcal{A}_j = \bigcup_{j \in \mathbb{N}} O_\delta^s(\hat{\omega}_j^t) \supset \bigcup_{j \in \mathbb{N}} O_\delta^T(\hat{\omega}_j^t) = \Omega^t$ and since

$$\mathbb{E}_{\mathbb{P}} \left[\int_t^T |g_r^{t,\omega}| dr + \Psi_*^{t,\omega} \right] < \infty \quad (7.13)$$

by (2.6), letting $\lambda \rightarrow \infty$, one can deduce from the dominated convergence theorem that

$$V_t^n(\omega) \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + \varepsilon.$$

Eventually, taking infimum over $\mathbb{P} \in \mathcal{P}(t, \omega)$ on the right-hand-side and then letting $\varepsilon \rightarrow 0$ yield (3.5). \square

Proof of Proposition 3.2: Let $0 \leq t \leq s \leq T$ and $\omega \in \Omega$. It suffices to show for a given $\mathbb{P} \in \mathcal{P}(t, \omega)$ that

$$\inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \gamma)] \geq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma < s\}} R^{t,\omega}(\tau, \gamma) + \mathbf{1}_{\{\tau \wedge \gamma \geq s\}} \left(\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right]. \quad (7.14)$$

Fix $\varepsilon > 0$. There exists a $\hat{\gamma} = \hat{\gamma}(\varepsilon) \in \mathcal{T}^t$ such that

$$\sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \hat{\gamma})] \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \gamma)] + \varepsilon/5. \quad (7.15)$$

1) Set $\hat{\gamma}' := \hat{\gamma} \vee s \in \mathcal{T}_s^t$. In the first step, we use a “dense” countable subset of \mathcal{T}^s and Proposition 1.2 to show that

$$\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \leq \operatorname{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t] + \frac{3}{5}\varepsilon, \quad \mathbb{P}\text{-a.s.} \quad (7.16)$$

As in the proof of [5, Proposition 4.1] (see part (2a) and (2c) therein), we can construct a dense countable subset Γ of \mathcal{T}^s in sense that for any $\delta > 0$, $\zeta \in \mathcal{T}^s$ and $\tilde{\mathbb{P}} \in \mathfrak{P}_s$,

$$\exists \{\zeta_n\}_{n \in \mathbb{N}} \subset \Gamma \text{ such that } \lim_{n \rightarrow \infty} \downarrow \varsigma_n(\hat{\omega}) = \zeta(\hat{\omega}), \quad \forall \hat{\omega} \in \Omega^s \text{ and that } \tilde{\mathbb{P}}\{\zeta_n \neq \zeta_n\} < \delta, \quad \forall n \in \mathbb{N}, \quad (7.17)$$

where $\zeta_n := \sum_{i=1}^{\lfloor 2^n T \rfloor} \mathbf{1}_{\{i2^{-n} \leq \zeta < (i+1)2^{-n}\}} \left(\frac{i+1}{2^n} \wedge T \right) \in \mathcal{T}^s$.

Since $\zeta(\Pi_s^t) \in \mathcal{T}_s^t$ for any $\zeta \in \mathcal{T}^s$ by (1.3), it holds except on a \mathbb{P} -null set \mathcal{N} that

$$\mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\zeta(\Pi_s^t), \hat{\gamma}') | \mathcal{F}_s^t] \leq \operatorname{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t], \quad \forall \zeta \in \Gamma. \quad (7.18)$$

By Proposition 1.1 (2), $\gamma_{\tilde{\omega}} := (\hat{\gamma}')^{s, \tilde{\omega}} \in \mathcal{T}^s$. In light of (1.9), there exists a \mathbb{P} -null set $\tilde{\mathcal{N}}$ such that for any $\tilde{\omega} \in \tilde{\mathcal{N}}^c$,

$$\mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\zeta(\Pi_s^t), \hat{\gamma}') | \mathcal{F}_s^t](\tilde{\omega}) = \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}}[R^{s, \omega \otimes_t \tilde{\omega}}(\zeta, \gamma_{\tilde{\omega}})] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr, \quad \forall \zeta \in \Gamma. \quad (7.19)$$

Here we used an analogy to (7.5) that $(R^{t, \omega}(\zeta(\Pi_s^t), \hat{\gamma}'))^{s, \tilde{\omega}} = R^{s, \omega \otimes_t \tilde{\omega}}(\zeta, \gamma_{\tilde{\omega}}) + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr$.

By (P2), there exist an extension $(\Omega^t, \mathcal{F}^t, \mathbb{P}')$ of $(\Omega^t, \mathcal{F}_T^t, \mathbb{P})$ and $\Omega' \in \mathcal{F}'$ with $\mathbb{P}'(\Omega') = 1$ such that for any $\tilde{\omega} \in \Omega'$, $\mathbb{P}^{s, \tilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \tilde{\omega})$. Let $\overline{\mathcal{N}}$ be the \mathcal{F}_T^t -measurable set containing $\mathcal{N} \cup \tilde{\mathcal{N}}$ and with $\mathbb{P}(\overline{\mathcal{N}}) = 0$.

Now, fix $\tilde{\omega} \in \Omega' \cap \overline{\mathcal{N}}^c \in \mathcal{F}'$. There exists a $\zeta_{\tilde{\omega}} \in \mathcal{T}^s$ such that

$$\sup_{\zeta \in \mathcal{T}^s} \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}}[R^{s, \omega \otimes_t \tilde{\omega}}(\zeta, \gamma_{\tilde{\omega}})] \leq \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}}[R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}})] + \varepsilon/5. \quad (7.20)$$

As $\mathbb{P}^{s, \tilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \tilde{\omega})$, (2.6) shows that

$$\mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} \left[\int_s^T |g_r^{s, \omega \otimes_t \tilde{\omega}}| dr + \Psi_*^{s, \omega \otimes_t \tilde{\omega}} \right] < \infty. \quad (7.21)$$

So for some $\delta_{\tilde{\omega}} > 0$,

$$\mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} \left[\mathbf{1}_A \left(\int_s^T |g_r^{s, \omega \otimes_t \tilde{\omega}}| dr + \Psi_*^{s, \omega \otimes_t \tilde{\omega}} \right) \right] < \varepsilon/5 \quad \text{for any } A \in \mathcal{F}_T^s \text{ with } \mathbb{P}^{s, \tilde{\omega}}(A) < \delta_{\tilde{\omega}}. \quad (7.22)$$

Applying (7.17) with $(\delta, \zeta, \tilde{\mathbb{P}}) = (\delta_{\tilde{\omega}}, \zeta_{\tilde{\omega}}, \mathbb{P}^{s, \tilde{\omega}})$, there exist $\{\zeta_{\tilde{\omega}}^k\}_{k \in \mathbb{N}} \subset \Gamma$ such that $\lim_{k \rightarrow \infty} \downarrow \zeta_{\tilde{\omega}}^k(\tilde{\omega}) = \zeta_{\tilde{\omega}}(\tilde{\omega})$, $\tilde{\omega} \in \Omega^s$ and that $\mathbb{P}^{s, \tilde{\omega}}\{\zeta_{\tilde{\omega}}^k \neq \zeta_{\tilde{\omega}}\} < \delta_{\tilde{\omega}}$, $\forall k \in \mathbb{N}$, where $\zeta_{\tilde{\omega}}^k := \sum_{i=\lfloor 2^k s \rfloor}^{\lfloor 2^k T \rfloor} \mathbf{1}_{\{i2^{-k} \leq \zeta_{\tilde{\omega}} < (i+1)2^{-k}\}} \left(\frac{i+1}{2^k} \wedge T \right) \in \mathcal{T}^s$.

Given $k \in \mathbb{N}$, (7.22) and (2.5) imply that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} \left[\left| R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}^k, \gamma_{\tilde{\omega}}) - R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}}) \right| \right] &= \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} \left[\mathbf{1}_{\{\zeta_{\tilde{\omega}}^k \neq \zeta_{\tilde{\omega}}\}} \left| R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}^k, \gamma_{\tilde{\omega}}) - R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}}) \right| \right] \\ &\leq 2 \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} \left[\mathbf{1}_{\{\zeta_{\tilde{\omega}}^k \neq \zeta_{\tilde{\omega}}\}} \left(\int_s^T |g_r^{s, \omega \otimes_t \tilde{\omega}}| dr + \Psi_*^{s, \omega \otimes_t \tilde{\omega}} \right) \right] < \frac{2}{5} \varepsilon, \end{aligned}$$

which together with (7.18) and (7.19) shows that

$$\mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} [R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}^k, \gamma_{\tilde{\omega}})] < \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} [R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}})] + \frac{2}{5} \varepsilon \leq \text{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t](\tilde{\omega}) - \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr + \frac{2}{5} \varepsilon.$$

As one can deduce from $\zeta_{\tilde{\omega}} = \lim_{k \rightarrow \infty} \downarrow \zeta_{\tilde{\omega}}^k$ and the continuity of L that

$$R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}}) \leq \lim_{k \rightarrow \infty} R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}^k, \gamma_{\tilde{\omega}}), \quad (7.23^*)$$

(2.5), (7.21), the dominated convergence theorem and (7.20) imply that

$$\begin{aligned} \overline{V}_s^{t, \omega}(\tilde{\omega}) &= \overline{V}_s(\omega \otimes_t \tilde{\omega}) \leq \sup_{\zeta \in \mathcal{T}^s} \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} [R^{s, \omega \otimes_t \tilde{\omega}}(\zeta, \gamma_{\tilde{\omega}})] \leq \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} [R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}})] + \varepsilon/5 \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{P}^{s, \tilde{\omega}}} [R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}^k, \gamma_{\tilde{\omega}})] + \varepsilon/5 \leq \text{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t](\tilde{\omega}) - \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr + \frac{3}{5} \varepsilon, \quad \forall \tilde{\omega} \in \Omega' \cap \overline{\mathcal{N}}^c, \end{aligned}$$

This shows $\Omega' \cap \overline{\mathcal{N}}^c \subset \overline{A} := \left\{ \overline{V}_s^{t, \omega} + \int_t^s g_r^{t, \omega} dr \leq \text{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t] + \frac{3}{5} \varepsilon \right\}$. As Remark 3.2 and Proposition 1.1

(1) imply that $\overline{V}_s^{t, \omega} + \int_t^s g_r^{t, \omega} dr = (\overline{V}_s + \int_t^s g_r dr)^{t, \omega} \in \mathcal{F}_s^t$, we see that $\overline{A} \in \mathcal{F}_s^t$ and thus $\mathbb{P}(\overline{A}) = \mathbb{P}'(\overline{A}) \geq \mathbb{P}'(\Omega' \cap \overline{\mathcal{N}}^c) = 1$. Therefore, (7.16) holds.

Moreover, one can find a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in \mathcal{T}_s^t such that

$$\text{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t] = \lim_{n \rightarrow \infty} \uparrow \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau_n, \hat{\gamma}') | \mathcal{F}_s^t], \quad \mathbb{P}\text{-a.s.} \quad (7.24^*)$$

2) Next, let $\tau \in \mathcal{T}^t$ and $n \in \mathbb{N}$. Since

$$\bar{\tau}_n := \mathbf{1}_{\{\tau \wedge \hat{\gamma} < s\}} \tau + \mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} \tau_n \quad (7.25^*)$$

defines an \mathbf{F}^t -stopping time, (7.16) and (3.3) show that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \hat{\gamma} < s\}} R^{t,\omega}(\tau, \hat{\gamma}) + \mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} \left(\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] \\ & \leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \hat{\gamma} < s\}} R^{t,\omega}(\bar{\tau}_n, \hat{\gamma}) + \mathbf{1}_{A_n \cap \{\tau \wedge \hat{\gamma} \geq s\}} \left(\mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau_n, \hat{\gamma}) | \mathcal{F}_s^t] + \frac{4}{5} \varepsilon \right) \right] + \alpha_n, \end{aligned} \quad (7.26)$$

where $A_n := \left\{ \text{esssup}_{\tau \in \mathcal{T}_s^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \hat{\gamma}) | \mathcal{F}_s^t] < \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau_n, \hat{\gamma}) | \mathcal{F}_s^t] + \varepsilon/5 \right\} \in \mathcal{F}_s^t$ and $\alpha_n := \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{A_n^c} \left(\int_t^T |g_r^{t,\omega}| dr + \Psi_*^{t,\omega} \right) \right]$.

Also, we can deduce from (2.5) that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{A_n \cap \{\tau \wedge \hat{\gamma} \geq s\}} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau_n, \hat{\gamma}) | \mathcal{F}_s^t] \right] = \mathbb{E}_{\mathbb{P}} \left[\mathbb{E}_{\mathbb{P}} [\mathbf{1}_{A_n \cap \{\tau \wedge \hat{\gamma} \geq s\}} R^{t,\omega}(\tau_n, \hat{\gamma}) | \mathcal{F}_s^t] \right] = \mathbb{E}_{\mathbb{P}} [\mathbf{1}_{A_n \cap \{\tau \wedge \hat{\gamma} \geq s\}} R^{t,\omega}(\tau_n, \hat{\gamma})] \\ & = \mathbb{E}_{\mathbb{P}} [\mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} R^{t,\omega}(\tau_n, \hat{\gamma}) - \mathbf{1}_{A_n^c \cap \{\tau \wedge \hat{\gamma} \geq s\}} R^{t,\omega}(\tau_n, \hat{\gamma})] \leq \mathbb{E}_{\mathbb{P}} [\mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} R^{t,\omega}(\bar{\tau}_n, \hat{\gamma})] + \alpha_n, \end{aligned}$$

which together with (7.26) and (7.15) leads to that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \hat{\gamma} < s\}} R^{t,\omega}(\tau, \hat{\gamma}) + \mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} \left(\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] & \leq \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\bar{\tau}_n, \hat{\gamma})] + 2\alpha_n + \frac{4}{5} \varepsilon \leq \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \hat{\gamma})] + 2\alpha_n + \frac{4}{5} \varepsilon \\ & \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \gamma)] + 2\alpha_n + \varepsilon. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \uparrow \mathbb{P}(A_n) = 1$ by (7.24), we see from (7.13) and the dominated convergence theorem that $\lim_{n \rightarrow \infty} \downarrow \alpha_n = 0$ and thus

$$\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \hat{\gamma} < s\}} R^{t,\omega}(\tau, \hat{\gamma}) + \mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} \left(\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}} [R^{t,\omega}(\tau, \gamma)] + \varepsilon, \quad \forall \tau \in \mathcal{T}^t. \quad (7.27)$$

Taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand-side and then letting $\varepsilon \rightarrow 0$ lead to (7.14). \square

Proof of Proposition 3.3: Let $n \in \mathbb{N}$, $t \in [0, T]$, $\alpha > 0$ and $\omega \in O_{\alpha}^t(\mathbf{0})$.

We fix $\mathbb{P} \in \mathcal{P}(t, \omega)$ and $\gamma, \tau \in \mathcal{T}^t$. Set $\{t_i^n\}_{i=0}^{2^n}$ as in (1.5) and define $\tau_n := \mathbf{1}_{\{\tau=t\}} t + \sum_{i=1}^{2^n} \mathbf{1}_{\{t_{i-1}^n < \tau \leq t_i^n\}} t_i^n \in \mathcal{T}^t(n)$. One can deduce that

$$\begin{aligned} R^{t,\omega}(\tau, \gamma) - R^{t,\omega}(\tau_n, \gamma) & = - \int_{\tau \wedge \gamma}^{\tau_n \wedge \gamma} g_r^{t,\omega} dr + \mathbf{1}_{\{\tau \leq \gamma\}} (L_{\tau}^{t,\omega} - \mathbf{1}_{\{\tau_n \leq \gamma\}} L_{\tau_n}^{t,\omega} - \mathbf{1}_{\{\gamma < \tau_n\}} U_{\gamma}^{t,\omega}) + \mathbf{1}_{\{\gamma < \tau\}} (U_{\gamma}^{t,\omega} - U_{\gamma}^{t,\omega}) \\ & = - \int_{\tau \wedge \gamma}^{\tau_n \wedge \gamma} g_r^{t,\omega} dr + \sum_{i=1}^{2^n} \left(\mathbf{1}_{\{t_{i-1}^n < \tau \leq t_i^n \leq \gamma\}} (L_{\tau}^{t,\omega} - L_{t_i^n}^{t,\omega}) + \mathbf{1}_{\{t_{i-1}^n < \tau \leq \gamma < t_i^n\}} (L_{\tau}^{t,\omega} - U_{\gamma}^{t,\omega}) \right). \end{aligned} \quad (7.28)$$

Given $i = 1, \dots, 2^n$, (1.6) shows that for any $\tilde{\omega} \in \{t_{i-1}^n < \tau \leq t_i^n \leq \gamma\}$

$$\begin{aligned} |L_{\tau}^{t,\omega}(\tilde{\omega}) - L_{t_i^n}^{t,\omega}(\tilde{\omega})| & = |L(\tau(\tilde{\omega}), \omega \otimes_t \tilde{\omega}) - L(t_i^n, \omega \otimes_t \tilde{\omega})| \leq \rho_0 \left((t_i^n - \tau(\tilde{\omega})) + \sup_{r \in [0, T]} |(\omega \otimes_t \tilde{\omega})(r \wedge \tau(\tilde{\omega})) - (\omega \otimes_t \tilde{\omega})(r \wedge t_i^n)| \right) \\ & \leq \rho_0 \left(2^{-n} + \sup_{r \in [\tau(\tilde{\omega}), t_i^n]} |\tilde{\omega}(r) - \tilde{\omega}(\tau(\tilde{\omega}))| \right) \leq \rho_0 \left(2^{-n} + \sup_{\tau(\tilde{\omega}) \leq r \leq (\tau(\tilde{\omega}) + 2^{-n}) \wedge T} |B_r^t(\tilde{\omega}) - B_{\tau}^t(\tilde{\omega})| \right). \end{aligned} \quad (7.29)$$

Similarly, it holds for any $\tilde{\omega} \in \{t_{i-1}^n < \tau \leq \gamma < t_i^n\}$ that

$$|U_{\tau}^{t,\omega} - U_{\gamma}^{t,\omega}|(\tilde{\omega}) \leq \rho_0 \left((\gamma(\tilde{\omega}) - \tau(\tilde{\omega})) + \sup_{r \in [\tau(\tilde{\omega}), \gamma(\tilde{\omega})]} |\tilde{\omega}(r) - \tilde{\omega}(\tau(\tilde{\omega}))| \right) \leq \rho_0 \left(2^{-n} + \sup_{\tau(\tilde{\omega}) \leq r \leq (\tau(\tilde{\omega}) + 2^{-n}) \wedge T} |B_r^t(\tilde{\omega}) - B_{\tau}^t(\tilde{\omega})| \right). \quad (7.30)$$

Moreover, another analogy to (7.29) shows that for any $(s, \tilde{\omega}) \in [t, T] \times \Omega^t$

$$|g_s^{t,\omega}(\tilde{\omega}) - g_t(\omega)| \leq |g(s, \omega \otimes_t \tilde{\omega}) - g(t, \omega)| \leq \rho_0 \left(s - t + \sup_{r \in [t, s]} |\tilde{\omega}(r)| \right) \leq \rho_0 \left(T - t + \sup_{r \in [t, T]} |B_r^t(\tilde{\omega}) - B_t^t(\tilde{\omega})| \right), \quad (7.31)$$

where we used the fact that $B_t^t = 0$ in the last inequality. Plugging (7.29)–(7.31) back into (7.28) leads to that

$$R^{t,\omega}(\tau, \gamma) - R^{t,\omega}(\tau_n, \gamma) \leq 2^{-n} \left[|g_t(\omega)| + \rho_0 \left(T - t + \sup_{r \in [t, T]} |B_r^t(\tilde{\omega}) - B_t^t(\tilde{\omega})| \right) \right] + \rho_0 \left(2^{-n} + \sup_{r \in [\tau, (\tau + 2^{-n}) \wedge T]} |B_r^t - B_\tau^t| \right).$$

Taking expectation $\mathbb{E}_{\mathbb{P}}[\cdot]$, we see from (3.6) that

$$\mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau, \gamma)] \leq \mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau_n, \gamma)] + I_\alpha^n \leq \sup_{\tau' \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau', \gamma)] + I_\alpha^n,$$

where $I_\alpha^n := \rho_\alpha(2^{-n}) + 2^{-n}(|g_t(\omega)| + \rho_\alpha(T - t))$. Taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand-side yields that

$$\sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau, \gamma)] \leq \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau, \gamma)] + I_\alpha^n.$$

Eventually, taking infimum over $\gamma \in \mathcal{T}^t$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$ leads to (3.7). \square

Proof of Proposition 3.4: Fix $n \in \mathbb{N} \cup \{\infty\}$, $\omega \in \Omega$ and set $\alpha := 1 + \|\omega\|_{0,T}$. Let $0 \leq t < s \leq T$ such that $\delta_{t,s} := (s - t) \vee \sup_{t \leq r < r' \leq s} |\omega(r') - \omega(r)| \leq T$.

1a) We first utilize Proposition 3.1 and (3.6) to show that

$$V_t^n(\omega) - V_s^n(\omega) \leq (s - t) \sup_{r \in [0, T]} |g_r(\omega)| + (2 + s - t) \rho_\alpha(\delta_{t,s}). \quad (7.32)$$

Let $\mathbb{P} \in \mathcal{P}(t, \omega)$. Applying (3.5) and taking $\gamma = s$ show that

$$\begin{aligned} V_t^n(\omega) - V_s^n(\omega) &\leq \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau < s\}} R^{t,\omega}(\tau, s) + \mathbf{1}_{\{\tau \geq s\}} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] - V_s^n(\omega) \\ &= \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau < s\}} L_\tau^{t,\omega} + \mathbf{1}_{\{\tau \geq s\}} (V_s^n)^{t,\omega} - V_s^n(\omega) + \int_t^{\tau \wedge s} g_r^{t,\omega} dr \right]. \end{aligned} \quad (7.33)$$

Then, let $\tau \in \mathcal{T}^t(n)$. For any $\tilde{\omega} \in \{\tau < s\}$, (1.6) implies that

$$\begin{aligned} |L_\tau^{t,\omega}(\tilde{\omega}) - L_s^{t,\omega}(\tilde{\omega})| &= |L(\tau(\tilde{\omega}), \omega \otimes_t \tilde{\omega}) - L(s, \omega \otimes_t \tilde{\omega})| \leq \rho_0 \left((s - t) + \sup_{r \in [t, T]} |\tilde{\omega}(r \wedge \tau(\tilde{\omega})) - \tilde{\omega}(r \wedge s)| \right) \\ &\leq \rho_0 \left((s - t) + \sup_{r \in [\tau(\tilde{\omega}), s]} |\tilde{\omega}(r) - \tilde{\omega}(\tau(\tilde{\omega}))| \right) \leq \rho_0 \left((s - t) + \sup_{r \in [\tau(\tilde{\omega}), (\tau(\tilde{\omega}) + s - t) \wedge T]} |B_r^t(\tilde{\omega}) - B_\tau^t(\tilde{\omega})| \right). \end{aligned} \quad (7.34)$$

Similarly, using (1.6) again and applying (1.8) with $\eta = g_t \in \mathcal{F}_t$ yields that for any $\tilde{\omega} \in \Omega^t$

$$\begin{aligned} \left| \int_t^{\tau(\tilde{\omega}) \wedge s} g_r^{t,\omega}(\tilde{\omega}) dr \right| &\leq \int_t^s |g_r^{t,\omega}(\tilde{\omega})| dr \leq \int_t^s (|g_t^{t,\omega}(\tilde{\omega})| + |g_r^{t,\omega}(\tilde{\omega}) - g_t^{t,\omega}(\tilde{\omega})|) dr \\ &\leq \int_t^s \left(|g_t(\omega)| + \rho_0 \left((s - t) + \sup_{r \in [t, s]} |B_r^t(\tilde{\omega}) - B_t^t(\tilde{\omega})| \right) \right) dr. \end{aligned} \quad (7.35)$$

Also, (3.4) shows that for any $\tilde{\omega} \in \Omega^t$

$$\begin{aligned} |V_s^n(\omega) - (V_s^n)^{t,\omega}(\tilde{\omega})| &= |V_s^n(\omega) - V^n(s, \omega \otimes_t \tilde{\omega})| \leq \rho_1(\|\omega - \omega \otimes_t \tilde{\omega}\|_{0,s}) = \rho_1 \left(\sup_{r \in [t, s]} |\omega(r) - \omega(t) - \tilde{\omega}(r)| \right) \\ &\leq \rho_1 \left(\sup_{r \in [t, s]} |\omega(r) - \omega(t)| + \sup_{r \in [t, s]} |\tilde{\omega}(r)| \right) \leq \rho_1 \left(\delta_{t,s} + \sup_{r \in [t, (t + \delta_{t,s}) \wedge T]} |B_r^t(\tilde{\omega}) - B_t^t(\tilde{\omega})| \right). \end{aligned} \quad (7.36)$$

Since $\|\omega\|_{0,t} \leq \|\omega\|_{0,T} < \alpha$, we can deduce from (7.34), (7.35), (3.3), (3.6) and (7.36) that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau < s\}} L_\tau^{t,\omega} + \mathbf{1}_{\{\tau \geq s\}} (V_s^n)^{t,\omega} - V_s^n(\omega) + \int_t^{\tau \wedge s} g_r^{t,\omega} dr \right] - (s - t) |g_t(\omega)| \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau < s\}} L_s^{t,\omega} + \mathbf{1}_{\{\tau \geq s\}} (V_s^n)^{t,\omega} - V_s^n(\omega) + \rho_1 \left((s - t) + \sup_{r \in [\tau, (\tau + s - t) \wedge T]} |B_r^t - B_\tau^t| \right) + (s - t) \rho_1 \left((s - t) + \sup_{r \in [t, s]} |B_r^t - B_t^t| \right) \right] \\ &\leq \mathbb{E}_{\mathbb{P}} [(V_s^n)^{t,\omega} - V_s^n(\omega)] + (1 + s - t) \rho_\alpha(s - t) \leq (2 + s - t) \rho_\alpha(\delta_{t,s}). \end{aligned}$$

Taking supremum over $\tau \in \mathcal{T}^t(n)$ on the left-hand-side, we obtain (7.32) from (7.33).

1b) Next, we show that for \bar{V} the inequality (7.32) can be strengthened as

$$|\bar{V}_s(\omega) - \bar{V}_t(\omega)| \leq (s-t) \sup_{r \in [0, T]} |g_r(\omega)| + (2+s-t)\rho_\alpha(\delta_{t,s}). \quad (7.37)$$

Fix $\varepsilon > 0$. We can find a $\mathbb{P} = \mathbb{P}(\varepsilon) \in \mathcal{P}(t, \omega)$ such that $\bar{V}_t(\omega) + \varepsilon/2 \geq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau, \gamma)]$. By (7.27), there exists some $\hat{\gamma} = \hat{\gamma}(\varepsilon) \in \mathcal{T}^t$ such that

$$\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \hat{\gamma} < s\}} R^{t,\omega}(\tau, \hat{\gamma}) + \mathbf{1}_{\{\tau \wedge \hat{\gamma} \geq s\}} \left(\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}[R^{t,\omega}(\tau, \gamma)] + \varepsilon/2, \quad \forall \tau \in \mathcal{T}^t.$$

In particular, taking $\tau = s$ on the left-hand-side gives that

$$\bar{V}_t(\omega) + \varepsilon \geq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\hat{\gamma} < s\}} R^{t,\omega}(s, \hat{\gamma}) + \mathbf{1}_{\{\hat{\gamma} \geq s\}} \left(\bar{V}_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] = \mathbb{E}_{\mathbb{P}} \left[\int_t^{\hat{\gamma} \wedge s} g_r^{t,\omega} dr + \mathbf{1}_{\{\hat{\gamma} < s\}} U_{\hat{\gamma}}^{t,\omega} + \mathbf{1}_{\{\hat{\gamma} \geq s\}} \bar{V}_s^{t,\omega} \right]. \quad (7.38)$$

An analogy to (7.34) and (7.35) shows that

$$\begin{aligned} |U_{\hat{\gamma}}^{t,\omega}(\tilde{\omega}) - U_s^{t,\omega}(\tilde{\omega})| &\leq \rho_0 \left((s-t) + \sup_{r \in [\hat{\gamma}(\tilde{\omega}), (\hat{\gamma}(\tilde{\omega}) + s - t) \wedge T]} |B_r^t(\tilde{\omega}) - B_{\hat{\gamma}}^t(\tilde{\omega})| \right), \quad \forall \tilde{\omega} \in \{\hat{\gamma} < s\} \quad \text{and} \\ \left| \int_t^{\hat{\gamma}(\tilde{\omega}) \wedge s} g_r^{t,\omega}(\tilde{\omega}) dr \right| &\leq (s-t) \left(|g_t(\omega)| + \rho_0 \left((s-t) + \sup_{r \in [t, s]} |B_r^t(\tilde{\omega}) - B_t^t(\tilde{\omega})| \right) \right), \quad \forall \tilde{\omega} \in \Omega^t. \end{aligned}$$

As $\|\omega\|_{0,t} \leq \|\omega\|_{0,T} < \alpha$, plugging them back to (7.38) and applying (7.36) with $n = \infty$, we can deduce from (3.6) and (3.3) that

$$\begin{aligned} \bar{V}_t(\omega) - \bar{V}_s(\omega) + \varepsilon + (s-t)|g_t(\omega)| &\geq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\hat{\gamma} < s\}} U_s^{t,\omega} + \mathbf{1}_{\{\hat{\gamma} \geq s\}} \bar{V}_s^{t,\omega} - \bar{V}_s(\omega) \right] - (1+s-t)\rho_\alpha(s-t) \\ &\geq \mathbb{E}_{\mathbb{P}} [\bar{V}_s^{t,\omega} - \bar{V}_s(\omega)] - (1+s-t)\rho_\alpha(s-t) \geq -(2+s-t)\rho_\alpha(\delta_{t,s}). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ and taking (7.32) with $n = \infty$ yield (7.37).

Since $\lim_{t \nearrow s} \delta_{t,s} = \lim_{s \searrow t} \delta_{t,s} = 0$, we can deduce from (7.32) and (7.37) that each path of V^n is both left-upper-semicontinuous and right-lower-semicontinuous, in particular, each path of \bar{V} is continuous.

2) Given $(t, \omega) \in [0, T] \times \Omega$, Remark 3.2, Proposition 1.1 (4) and Part 1 show that $\bar{V}^{t,\omega}$ is an \mathbf{F}^t -adapted process with all continuous paths. For any $\mathbb{P} \in \mathcal{P}(t, \omega)$, (3.3) and (2.6) imply that $\mathbb{E}_{\mathbb{P}}[\bar{V}_*^{t,\omega}] \leq \mathbb{E}_{\mathbb{P}}[\Psi_*^{t,\omega}] < \infty$. So $\bar{V}^{t,\omega} \in \mathcal{S}(\mathbf{F}^t, \mathbb{P})$. \square

7.3 Proofs of the results in Section 4

Proof of (4.1): Fix $n \in \mathbb{N} \cup \{\infty\}$ and $\tau \in \mathcal{T}$. We let $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$. Since $V_\tau^n \in \mathcal{F}_T$ and $\int_0^\tau g_r dr \in \mathcal{F}_T$ by Remark 3.2, Proposition 1.1 (1) shows that both $(V_\tau^n)^{t,\omega}$ and $(\int_0^\tau g_r dr)^{t,\omega}$ belong to \mathcal{F}_T^t .

1) If $\hat{t} := \tau(\omega) \leq t$, Proposition 1.1 (3) shows that $\tau(\omega \otimes_t \Omega^t) \equiv \hat{t}$. Applying (1.8) to $\eta = V_{\hat{t}}^n \in \mathcal{F}_{\hat{t}} \subset \mathcal{F}_t$ and to $\eta = \int_0^{\hat{t}} g_r dr \in \mathcal{F}_{\hat{t}} \subset \mathcal{F}_t$ yields that for any $\tilde{\omega} \in \Omega^t$

$$(V_\tau^n)^{t,\omega}(\tilde{\omega}) = V^n(\tau(\omega \otimes_t \tilde{\omega}), \omega \otimes_t \tilde{\omega}) = V^n(\hat{t}, \omega \otimes_t \tilde{\omega}) = V^n(\hat{t}, \omega), \quad (7.39)$$

and $(\int_0^\tau g_r dr)^{t,\omega}(\tilde{\omega}) = \int_0^{\tau(\omega \otimes_t \tilde{\omega})} g_r(\omega \otimes_t \tilde{\omega}) dr = \int_0^{\hat{t}} g_r(\omega \otimes_t \tilde{\omega}) dr = \int_0^{\hat{t}} g_r(\omega) dr$. Both only depend on ω .

2) Next, suppose that $\tau > t$. Proposition 1.1 (3) also shows that $\tau(\omega \otimes_t \tilde{\omega}) > t$, $\forall \tilde{\omega} \in \Omega^t$ and that $\zeta := \tau^{t,\omega}$ is a \mathcal{T}^t -stopping time. It follows that $(V_\tau^n)^{t,\omega}(\tilde{\omega}) = V^n(\tau(\omega \otimes_t \tilde{\omega}), \omega \otimes_t \tilde{\omega}) = V^n(\tau^{t,\omega}(\tilde{\omega}), \omega \otimes_t \tilde{\omega}) = (V^n)^{t,\omega}(\zeta(\tilde{\omega}), \tilde{\omega})$, $\forall \tilde{\omega} \in \Omega^t$. By the first equality of (7.4), we also have $(\int_0^\tau g_r dr)^{t,\omega}(\tilde{\omega}) = \int_0^{\tau(\omega \otimes_t \tilde{\omega})} g_r(\omega \otimes_t \tilde{\omega}) dr = \int_0^t g_r(\omega) dr + \int_t^{\zeta(\tilde{\omega})} g_r^{t,\omega}(\tilde{\omega}) dr$. Then (3.3) and (2.6) imply that

$$\mathbb{E}_{\mathbb{P}} \left[|(V_\tau^n)^{t,\omega}| + \left| \left(\int_0^\tau g_r dr \right)^{t,\omega} \right| \right] \leq \mathbb{E}_{\mathbb{P}} \left[|(V^n)_\zeta^{t,\omega}| + \int_t^\zeta |g_r^{t,\omega}| dr \right] + \int_0^t |g_r(\omega)| dr \leq \mathbb{E}_{\mathbb{P}} \left[\Psi_*^{t,\omega} + \int_t^T |g_r^{t,\omega}| dr \right] + \int_0^t |g_r(\omega)| dr < \infty.$$

Proof of Theorem 4.1: Define $\bar{\Upsilon}_t := \bar{V}_t + \int_0^t g_r dr$, $t \in [0, T]$ as in Lemma A.1.

Given $(t, \omega) \in [0, T] \times \Omega$ and $n \in \mathbb{N}$, since Remark 3.2, Proposition 1.1 (4) and Proposition 3.4 show that $(V^n)^{t, \omega} - L^{t, \omega}$ is an \mathbf{F}^t -adapted process with left-upper-semicontinuous paths and that $\bar{V}^{t, \omega} - L^{t, \omega}$ is an \mathbf{F}^t -adapted process with all continuous paths, we can deduce from (3.2) that

$$\tau_{(t, \omega)}^{n, \delta} := \inf \{s \in [t, T] : (V^n)_s^{t, \omega} < L_s^{t, \omega} + \delta\}, \quad \forall \delta > 0$$

are all \mathbf{F}^t -optional times and that

$$\tau_{(t, \omega)}^* := \inf \{s \in [t, T] : \bar{V}_s^{t, \omega} = L_s^{t, \omega}\} = \inf \{s \in [t, T] : \bar{V}_s^{t, \omega} \leq L_s^{t, \omega}\}$$

is an \mathbf{F}^t -stopping time.

1) Let $(t, \omega) \in [0, T] \times \Omega$ and $\gamma \in \mathcal{T}^t$. Since $\gamma(\Pi_t^0) \in \mathcal{T}_t$ by (1.3), Taking $t' = t$ and $\zeta = \gamma(\Pi_t^0)$ in (A.1) of Lemma A.1 shows that

$$\bar{V}_t(\omega) + \int_0^t g_r(\omega) dr = \bar{\Upsilon}_t(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[\left(\bar{\Upsilon}_{(\tau_{(t, \omega)}^*, (\Pi_t^0) \wedge \gamma(\Pi_t^0)) \vee t} \right)^{t, \omega} \right]. \quad (7.40)$$

For any $\tilde{\omega} \in \Omega^t$, (3.3) and the first equality in (7.4) imply that

$$\begin{aligned} \left(\bar{\Upsilon}_{(\tau_{(t, \omega)}^*, (\Pi_t^0) \wedge \gamma(\Pi_t^0)) \vee t} \right)^{t, \omega}(\tilde{\omega}) &= \bar{\Upsilon} \left(\left(\tau_{(t, \omega)}^* (\Pi_t^0(\omega \otimes_t \tilde{\omega})) \wedge \gamma(\Pi_t^0(\omega \otimes_t \tilde{\omega})) \right) \vee t, \omega \otimes_t \tilde{\omega} \right) = \bar{\Upsilon}(\tau_{(t, \omega)}^*(\tilde{\omega}) \wedge \gamma(\tilde{\omega}), \omega \otimes_t \tilde{\omega}) \\ &= \bar{V}^{t, \omega}(\tau_{(t, \omega)}^*(\tilde{\omega}) \wedge \gamma(\tilde{\omega}), \tilde{\omega}) + \int_0^{\tau_{(t, \omega)}^*(\tilde{\omega}) \wedge \gamma(\tilde{\omega})} g_r(\omega \otimes_t \tilde{\omega}) dr \\ &\leq \mathbf{1}_{\{\tau_{(t, \omega)}^*(\tilde{\omega}) \leq \gamma(\tilde{\omega})\}} L^{t, \omega}(\tau_{(t, \omega)}^*(\tilde{\omega}), \tilde{\omega}) + \mathbf{1}_{\{\gamma(\tilde{\omega}) < \tau_{(t, \omega)}^*(\tilde{\omega})\}} U^{t, \omega}(\gamma(\tilde{\omega}), \tilde{\omega}) + \int_0^t g_r(\omega) dr + \int_t^{\tau_{(t, \omega)}^*(\tilde{\omega}) \wedge \gamma(\tilde{\omega})} g_r^{t, \omega}(\tilde{\omega}) dr \\ &= (R^{t, \omega}(\tau_{(t, \omega)}^*, \gamma))(\tilde{\omega}) + \int_0^t g_r(\omega) dr. \end{aligned}$$

Plugging this into (7.40) yields that $\bar{V}_t(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau_{(t, \omega)}^*, \gamma)]$. Taking infimum over $\gamma \in \mathcal{T}^t$ leads to that

$$\bar{V}_t(\omega) \leq \inf_{\gamma \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau_{(t, \omega)}^*, \gamma)] \leq \sup_{\tau \in \mathcal{T}^t} \inf_{\gamma \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [R^{t, \omega}(\tau, \gamma)] = \underline{V}_t(\omega) \leq \bar{V}_t(\omega), \quad \text{proving (4.3).}$$

2) Let $\zeta \in \mathcal{T}$ and $(t, \omega) \in [0, T] \times \Omega$. If $\hat{t} := \tau_*(\omega) \wedge \zeta(\omega) \leq t$, similar to (7.39), we can deduce from Proposition 1.1 (3), the \mathbf{F} -adaptedness of Υ by Remark 3.2 as well as (1.8) that $(\Upsilon_{\tau_* \wedge \zeta})^{t, \omega}(\tilde{\omega}) = \Upsilon(\hat{t}, \omega)$, $\forall \tilde{\omega} \in \Omega^t$. Then

$$\underline{\mathcal{E}}_t[\Upsilon_{\tau_* \wedge \zeta}](\omega) = \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [(\Upsilon_{\tau_* \wedge \zeta})^{t, \omega}] = \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [\Upsilon(\hat{t}, \omega)] = \Upsilon(\hat{t}, \omega) = \Upsilon(\tau_*(\omega) \wedge \zeta(\omega) \wedge t, \omega). \quad (7.41)$$

On the other hand, if $\tau_*(\omega) \wedge \zeta(\omega) > t$, applying Proposition 1.1 (3) once again shows that $\omega \otimes_t \Omega^t \subset \{\tau_* \wedge \zeta > t\}$. So it holds for any $\tilde{\omega} \in \Omega^t$ that $(\Upsilon_{\tau_* \wedge \zeta})^{t, \omega}(\tilde{\omega}) = \Upsilon_{\tau_* \wedge \zeta}(\omega \otimes_t \tilde{\omega}) = \Upsilon_{(\tau_* \wedge \zeta) \vee t}(\omega \otimes_t \tilde{\omega}) = (\Upsilon_{(\tau_* \wedge \zeta) \vee t})^{t, \omega}(\tilde{\omega})$. As $\tau_* = \tau_{(0, \mathbf{0})}^* = \tau_{(0, \omega)}^*$, taking $t' = 0$ in (A.1) yields that

$$\Upsilon_{\tau_* \wedge \zeta \wedge t}(\omega) = \Upsilon_t(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [(\Upsilon_{(\tau_* \wedge \zeta) \vee t})^{t, \omega}] = \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} [(\Upsilon_{\tau_* \wedge \zeta})^{t, \omega}] = \underline{\mathcal{E}}_t[\Upsilon_{\tau_* \wedge \zeta}](\omega),$$

which together with (7.41) proves (4.4). \square

7.4 Proof of Proposition 5.1

For any $\alpha, \delta \in (0, \infty)$, we define $\Phi(\alpha, \delta) := \varrho_0(\delta + \delta^{1/4}) + \kappa(1 + 2^{\varpi-1} \delta^{\varpi}) \varphi_1(\alpha) \delta^{1/4} + \kappa 2^{\varpi-1} \varphi_{\varpi+1}(\alpha) \delta^{\varpi/2+1/4}$.

1) we first show that the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ satisfies (P1) and (P2).

Let $(t, \omega) \in [0, T] \times \Omega$ and $\mu \in \mathcal{U}_t$. We set $(\mathbb{P}, \mathbf{p}, \mathcal{X}) := (\mathbb{P}^{t, \omega, \mu}, \mathbf{p}^{t, \omega, \mu}, X^{t, \omega, \mu})$. Given $\tilde{\omega} \in \Omega^t$, (2.4) shows that

$$|\Psi_r^{t, \mathbf{0}}(\mathcal{X}(\tilde{\omega})) - \Psi_r(\mathbf{0})| = |\Psi_r(\mathbf{0} \otimes_t \mathcal{X}(\tilde{\omega})) - \Psi_r(\mathbf{0})| \leq \varrho_0(\|\mathbf{0} \otimes_t \mathcal{X}(\tilde{\omega})\|_{0, r}) \leq \kappa(1 + \|\mathcal{X}(\tilde{\omega})\|_{t, r}^{\varpi}), \quad \forall r \in [t, T]. \quad (7.42)$$

It follows that $\Psi_*^{t,0}(\mathcal{X}(\tilde{\omega})) = \sup_{r \in [t, T]} |\Psi_r^{t,0}(\mathcal{X}(\tilde{\omega}))| \leq \kappa(1 + \|\mathcal{X}(\tilde{\omega})\|_{t, T}^{\varpi}) + M_0^\Psi$, where $M_0^\Psi := \sup_{r \in [t, T]} |\Psi_r(\mathbf{0})| < \infty$ by the continuity of path $\Psi(\mathbf{0})$. Since $\Psi^{t,0}$ is an \mathbf{F}^t -adapted process by Proposition 1.1 (4), applying (5.3) yields that

$$\mathbb{E}_{\mathbb{P}}[\Psi_*^{t,0}] = \mathbb{E}_{\mathbb{P}}[\Psi_*^{t,0}] = \mathbb{E}_t[\Psi_*^{t,0}(\mathcal{X})] \leq \kappa(1 + \mathbb{E}_t[\|\mathcal{X}\|_{t, T}^{\varpi}]) + M_0^\Psi \leq \kappa(1 + \varphi_{\varpi}(\|\omega\|_{0, t}) T^{\varpi/2}) + M_0^\Psi < \infty.$$

Namely, $\Psi^{t,0} \in \mathbb{S}(\mathbf{F}^t, \mathbb{P})$. Similar to (7.42), one can deduce from (1.6) that $|g_r^{t,0}(\mathcal{X}(\tilde{\omega})) - g_r(\mathbf{0})| \leq \kappa(1 + \|\mathcal{X}(\tilde{\omega})\|_{t, r}^{\varpi})$ for any $r \in [t, T]$. Then Fubini's Theorem and (5.3) imply that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \int_t^T |g_r^{t,0}| dr &= \mathbb{E}_{\mathbb{P}} \int_t^T |g_r^{t,0}| dr = \mathbb{E}_t \int_t^T |g_r^{t,0}(\mathcal{X})| dr \leq \kappa \int_t^T (1 + \mathbb{E}_t[\|\mathcal{X}\|_{t, T}^{\varpi}]) dr + \int_t^T |g_r(\mathbf{0})| dr \\ &\leq \kappa T(1 + \varphi_{\varpi}(\|\omega\|_{0, t}) T^{\varpi/2}) + \int_t^T |g_r(\mathbf{0})| dr < \infty. \quad \text{Hence } \mathbb{P} \in \widehat{\mathfrak{P}}_t. \end{aligned} \quad (7.43)$$

For any $t \in [0, T]$ and $\omega_1, \omega_2 \in \Omega$ with $\omega_1|_{[0, t]} = \omega_2|_{[0, t]}$, since the SDE (5.1) depends only on $\omega|_{[0, t]}$ for a given path $\omega \in \Omega$, we see that $X^{t, \omega_1, \mu} = X^{t, \omega_2, \mu}$ and thus $\mathbb{P}^{t, \omega_1, \mu} = \mathbb{P}^{t, \omega_2, \mu}$ for any $\mu \in \mathcal{U}_t$. It follows that $\mathcal{P}(t, \omega_1) = \mathcal{P}(t, \omega_2)$. So Assumption (P1) is satisfied. Also, Proposition 6.3 of [5] has already shown that the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ satisfies (P2).

2) The verification that the probability class $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ satisfies (P3) is relatively lengthy. We split it into several steps.

2a) Let us first quote some knowledge on the inverse mapping of $X^{t, \omega, \mu}$ from [5], which has already verified (P3) (i), (ii) for $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$.

Given $(t, \omega) \in [0, T] \times \Omega$ and $\mu \in \mathcal{U}_t$, according to [5] (see the context around (7.62) and (7.63) therein), there exists an \mathbf{F}^t -progressively measurable process $W^{t, \omega, \mu}$ such that for all $\tilde{\omega} \in \Omega^t$ except on a \mathbb{P}_0^t -null set $\mathcal{N}_{t, \omega, \mu}^c$

$$B_s^t(\tilde{\omega}) = W_s^{t, \omega, \mu}(X^{t, \omega, \mu}(\tilde{\omega})), \quad \forall s \in [t, T],$$

and that the $\mathbf{p}^{t, \omega, \mu}$ probability of set $A_{t, \omega, \mu} := \{\tilde{\omega}' \in \Omega^t : \mathcal{N}_{t, \omega, \mu}^c \cap (X^{t, \omega, \mu})^{-1}(\tilde{\omega}') \neq \emptyset\}$ is 1, i.e., $A_{t, \omega, \mu}^c \in \mathcal{N}^{\mathbf{p}^{t, \omega, \mu}} := \{A \in \mathcal{G}_T^{X^{t, \omega, \mu}} : \mathbf{p}^{t, \omega, \mu}(A) = 0\}$. For any $r \in [t, T]$, (5.4) and Lemma A.3 (2) of [5] show that $\mathfrak{F}_r^{t, \omega, \mu} := \sigma(\mathcal{F}_r^t \cup \mathcal{N}^{\mathbf{p}^{t, \omega, \mu}}) \subset \mathcal{G}_r^{X^{t, \omega, \mu}}$.

We see from the context around (7.67)–(7.69) of [5] that $\widetilde{W}_r^{t, \omega, \mu}(\tilde{\omega}) := \mathbf{1}_{\{\tilde{\omega} \in A_{t, \omega, \mu}\}} W_r^{t, \omega, \mu}(\tilde{\omega})$, $(r, \tilde{\omega}) \in [t, T] \times \Omega^t$ is an $\{\mathfrak{F}_r^{t, \omega, \mu}\}_{r \in [t, T]}$ -adapted process such that all its paths belong to Ω^t , that

$$\tilde{\omega} = B^t(\tilde{\omega}) = W^{t, \omega, \mu}(X^{t, \omega, \mu}(\tilde{\omega})) = \widetilde{W}^{t, \omega, \mu}(X^{t, \omega, \mu}(\tilde{\omega})), \quad \forall \tilde{\omega} \in \mathcal{N}_{t, \omega, \mu}^c, \quad (7.44)$$

and that

$$(\widetilde{W}^{t, \omega, \mu})^{-1}(A') \in \mathfrak{F}_r^{t, \omega, \mu}, \quad \forall A' \in \overline{\mathcal{F}}_r^t, \quad \forall r \in [t, T]. \quad (7.45)$$

Fix $0 \leq t < s \leq T$, $\omega \in \Omega$ and $\mu \in \mathcal{U}_t$, $\delta \in \mathbb{Q}_+$ and $\lambda \in \mathbb{N}$. We consider a \mathcal{F}_s^t -partition $\{\mathcal{A}_j\}_{j=0}^\lambda$ of Ω^t such that for $j=1, \dots, \lambda$, $\mathcal{A}_j \subset O_{\delta_j}^s(\tilde{\omega}_j)$ for some $\delta_j \in ((0, \delta] \cap \mathbb{Q}) \cup \{\delta\}$ and $\tilde{\omega}_j \in \Omega^t$, and let $\{\mu_j^j\}_{j=1}^\lambda \subset \mathcal{U}_s$. We will simply set

$$(\mathbb{P}, \mathbf{p}, \mathcal{X}, \mathcal{W}, \mathfrak{F}, \cdot) := (\mathbb{P}^{t, \omega, \mu}, \mathbf{p}^{t, \omega, \mu}, X^{t, \omega, \mu}, \widetilde{W}^{t, \omega, \mu}, \mathfrak{F}^{t, \omega, \mu}). \quad (7.46)$$

Given $j=1, \dots, \lambda$, (5.4) shows that $\mathcal{A}_j^{\mathcal{X}} := \mathcal{X}^{-1}(\mathcal{A}_j) \in \overline{\mathcal{F}}_s^t$. So there exists an $A_j \in \mathcal{F}_s^t$ such that $\mathcal{A}_j^{\mathcal{X}} \Delta A_j \in \overline{\mathcal{N}}^t$ (see e.g. Problem 2.7.3 of [36]). Following similar arguments to those used in the proof of Proposition 6.3 of [5], one can show that

(u1) The set $\tilde{A}_j := A_j \setminus \bigcup_{j' < j} A_{j'} \in \mathcal{F}_s^t$ satisfies $\mathcal{A}_j^{\mathcal{X}} \Delta \tilde{A}_j \in \overline{\mathcal{N}}^t$ (see (7.70) of [5]).

(u2) The pasted control $\hat{\mu}_r(\tilde{\omega}) := \mathbf{1}_{\{r \in [t, s)\}} \mu_r(\tilde{\omega}) + \mathbf{1}_{\{r \in [s, T]\}} \left(\mathbf{1}_{\{\tilde{\omega} \in \tilde{A}_0\}} \mu_r(\tilde{\omega}) + \sum_{j=1}^\lambda \mathbf{1}_{\{\tilde{\omega} \in \tilde{A}_j\}} \mu_r^j(\Pi_s^t(\tilde{\omega})) \right)$, $\forall (r, \tilde{\omega}) \in [t, T] \times \Omega^t$ belongs to \mathcal{U}_t , where $\tilde{A}_0 := \left(\bigcup_{j=1}^\lambda \tilde{A}_j \right)^c \in \mathcal{F}_s^t$ (see (7.71) of [5]). Set

$$(\hat{\mathbb{P}}, \hat{\mathbf{p}}, \hat{\mathcal{X}}, \hat{\mathcal{W}}, \hat{\mathfrak{F}}, \hat{\mathcal{N}}) := (\mathbb{P}^{t, \omega, \hat{\mu}}, \mathbf{p}^{t, \omega, \hat{\mu}}, X^{t, \omega, \hat{\mu}}, \widetilde{W}^{t, \omega, \hat{\mu}}, \mathfrak{F}^{t, \omega, \hat{\mu}}, \mathcal{N}_{t, \omega, \hat{\mu}}).$$

(u3) There exists a \mathbb{P}_0^t -null set $\tilde{\mathcal{N}}_j$ such that for any $\tilde{\omega} \in \tilde{A}_j \cap \tilde{\mathcal{N}}_j^c$,

$$\mathcal{N}_{\tilde{\omega}} := \{\tilde{\omega} \in \Omega^s : \hat{\mathcal{X}}_r(\tilde{\omega} \otimes_s \tilde{\omega}) \neq (\mathcal{X}(\tilde{\omega}) \otimes_s X^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega}), \mu^j}(\tilde{\omega}))(r) \text{ for some } r \in [t, T]\} \text{ belongs to } \overline{\mathcal{N}}^s \text{ (see (7.78) of [5])}.$$

(u4) For any $A \in \mathcal{F}_s^t$, $\mathcal{X}^{-1}(A)\Delta\hat{\mathcal{X}}^{-1}(A) \in \overline{\mathcal{N}}^t$ (see (7.74) of [5]).

Also, analogous to part (2b) of [5, Proposition 6.3], we can use the uniqueness of controlled SDE (5.1) to show that the equality $\hat{\mu} = \mu$ over $([t, s] \times \Omega^t) \cup ([s, T] \times \tilde{A}_0)$ implies the equality $\hat{\mathcal{X}} = \mathcal{X}$ over $([t, s] \times \Omega^t) \cup ([s, T] \times \tilde{A}_0)$, and thus that $\hat{\mathbb{P}}$ satisfies (P3) (i), (ii).

2b) To show that $\hat{\mathbb{P}}$ satisfies (2.8), we make some technical setting and preparation first.

Proposition 1.1 (4) shows that $\mathcal{Y}_r^1 := g_r^{t, \omega}$, $\mathcal{Y}_r^2 := L_r^{t, \omega}$ and $\mathcal{Y}_r^3 := U_r^{t, \omega}$, $r \in [t, T]$ are three \mathbf{F}^t -adapted processes with all continuous paths. For $\ell = 1, 2, 3$, (5.4) implies that $\mathcal{Y}^\ell(\hat{\mathcal{X}})$ is an $\overline{\mathbf{F}}^t$ -adapted process with all continuous paths. Applying Lemma A.2 (3) of [5] with $(\mathbb{P}, X) = (\mathbb{P}_0^t, B^t)$ shows that $\mathcal{Y}^\ell(\hat{\mathcal{X}})$ has an $(\mathbf{F}^t, \mathbb{P}_0^t)$ -version \mathcal{Y}^ℓ . More precisely, \mathcal{Y}^ℓ 's are \mathbf{F}^t -progressively measurable processes such that

$$\mathcal{N}_R := \bigcup_{\ell=1}^3 \{\tilde{\omega} \in \Omega^t : \mathcal{Y}_r^\ell(\tilde{\omega}) \neq \mathcal{Y}_r^\ell(\hat{\mathcal{X}}(\tilde{\omega})) \text{ for some } r \in [t, T]\} \in \overline{\mathcal{N}}^t. \quad (7.47)$$

By Lemma 1.2, it holds for all $\tilde{\omega} \in \Omega^t$ except on an $\tilde{\mathcal{N}}_R \in \overline{\mathcal{N}}^t$ that $(\mathcal{N}_R \cup \tilde{\mathcal{N}}_R)^{s, \tilde{\omega}} \in \overline{\mathcal{N}}^s$.

We see from Proposition 1.1 (4) that the random variables

$$\xi_m := \sup_{t' \in [t, T]} \int_{t'}^{(t' + 2^{-m}) \wedge T} |g_r^{t, \omega}| dr, \quad \forall m \in \mathbb{N} \quad (7.48)$$

are \mathcal{F}_T^t -measurable. Since $\lim_{m \rightarrow \infty} \downarrow \xi_m = 0$, (2.6) and the dominated convergence theorem show that $\lim_{m \rightarrow \infty} \downarrow \mathbb{E}_{\hat{\mathbb{P}}}[\xi_m] = 0$. So there exists $\mathbf{m} \in \mathbb{N}$ such that $\mathbb{E}_{\hat{\mathbb{P}}}[\xi_{\mathbf{m}}] \leq \delta/2$ and $\Phi(\|\omega\|_{0, t}, 2^{-\mathbf{m}}) \leq \delta/2$. Set $\mathbf{a} := 2^{-\mathbf{m}}$.

Now, fix $n \in \mathbb{N} \cup \{\infty\}$, $\varphi \in \mathcal{T}^s$ and let $j = 1, \dots, \lambda$. We set

$$(\mathbb{P}^j, \mathbf{p}^j, \mathcal{X}^j, \mathcal{W}^j, \mathfrak{F}^j, \mathcal{N}_{\mathcal{X}^j}) := (\mathbb{P}^{s, \omega \otimes_t \tilde{\omega}_j, \mu^j}, \mathbf{p}^{s, \omega \otimes_t \tilde{\omega}_j, \mu^j}, X^{s, \omega \otimes_t \tilde{\omega}_j, \mu^j}, \tilde{W}^{s, \omega \otimes_t \tilde{\omega}_j, \mu^j}, \mathfrak{F}^{s, \omega \otimes_t \tilde{\omega}_j, \mu^j}, \mathcal{N}_{s, \omega \otimes_t \tilde{\omega}_j, \mu^j})$$

and define

$$\varphi_j := \varphi(\mathcal{X}^j) \in \overline{\mathcal{T}}^s, \quad \nu_j := \varphi_j(\Pi_s^t) \in \overline{\mathcal{T}}_s^t, \quad \hat{\gamma}_j := \nu_j(\widehat{\mathcal{W}}), \quad (7.49^*)$$

where $\hat{\gamma}_j$ is a $\hat{\mathfrak{F}}$ -stopping time that takes values in $[s, T]$.

Given $i = 0, \dots, 2^{\mathbf{m}}$, we set $s_i := s \vee (i2^{-\mathbf{m}}T)$ and $D_i := \{s_{i-1} < \hat{\gamma}_j \leq s_i\} \in \hat{\mathfrak{F}}_{s_i}$ with $s_{-1} := -1$. By e.g. Problem 2.7.3 of [36], there exists an $\tilde{D}_i \in \mathcal{F}_{s_i}^t$ such that $D_i \Delta \tilde{D}_i \in \mathcal{N}^{\hat{\mathbb{P}}}$. Define $\mathcal{D}_i := \tilde{D}_i \setminus \bigcup_{i' < i} \tilde{D}_{i'} \in \mathcal{F}_{s_i}^t$ and $\overline{\mathcal{D}} := \bigcup_{i=0}^{2^{\mathbf{m}}} \mathcal{D}_i = \bigcup_{i=0}^{2^{\mathbf{m}}} \tilde{D}_i \in \mathcal{F}_T^t$. Then $\gamma'_j := \sum_{i=0}^{2^{\mathbf{m}}} \mathbf{1}_{D_i} s_i$ is a $\hat{\mathfrak{F}}$ -stopping time while $\gamma_j := \sum_{i=0}^{2^{\mathbf{m}}} \mathbf{1}_{\mathcal{D}_i} s_i + \mathbf{1}_{\overline{\mathcal{D}}} T$ defines an \mathcal{T}_s^t -stopping time. Clearly, γ'_j coincides with γ_j over $\bigcup_{i=1}^{2^{\mathbf{m}}} (D_i \cap \mathcal{D}_i)$, whose complement $\bigcup_{i=1}^{2^{\mathbf{m}}} (D_i \setminus \mathcal{D}_i)$ belongs to $\mathcal{N}^{\hat{\mathbb{P}}}$ because

$$D_i \setminus \mathcal{D}_i = D_i \cap \left[(\tilde{D}_i)^c \cup \left(\bigcup_{i' < i} \tilde{D}_{i'} \right) \right] = (D_i \setminus \tilde{D}_i) \cup \left(\bigcup_{i' < i} (\tilde{D}_{i'} \cap D_i) \right) \subset (D_i \Delta \tilde{D}_i) \cup \left(\bigcup_{i' < i} (\tilde{D}_{i'} \cap D_i^c) \right) \subset \bigcup_{i' \leq i} (D_{i'} \Delta \tilde{D}_{i'}) \in \mathcal{N}^{\hat{\mathbb{P}}}.$$

for $i = 1, \dots, 2^{\mathbf{m}}$. To wit, we have

$$\gamma'_j = \gamma_j, \quad \hat{\mathbf{p}} - a.s. \quad (7.50)$$

2c) Now, fix $A \in \mathcal{F}_s^t$, $\tau \in \mathcal{T}_s^t(n)$ and set $\hat{\tau} := \tau(\hat{\mathcal{X}})$. We show an auxiliary inequality:

$$\sum_{j=1}^{\lambda} \mathbb{E}_{\hat{\mathbb{P}}}[\mathbf{1}_{A \cap \mathcal{A}_j} R^{t, \omega}(\tau, \gamma_j)] \leq \sum_{j=1}^{\lambda} \mathbb{E}_t[\mathbf{1}_{\hat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \Xi_j] + \delta, \quad (7.51)$$

where $\Xi_j := \int_t^{\hat{\tau} \wedge \nu_j} \mathcal{Y}_r^1 dr + \mathbf{1}_{\{\hat{\tau} \leq \nu_j\}} \mathcal{Y}_{\hat{\tau}}^2 + \mathbf{1}_{\{\nu_j < \hat{\tau}\}} \mathcal{Y}_{\nu_j}^3$.

For any $r \in [s, T]$, an analogy to (A.19) shows that $\{\hat{\tau} \leq r\} = \hat{\mathcal{X}}^{-1}(\{\tau \leq r\}) \in \overline{\mathcal{F}}_r^t$. So $\hat{\tau} \in \overline{\mathcal{T}}_s^t$. By Lemma 2.5 (3) in the ArXiv version of [5], it holds for all $\tilde{\omega} \in \Omega^t$ except on a $\mathcal{N}_{\tau} \in \overline{\mathcal{N}}^t$ that $\hat{\tau}^{s, \tilde{\omega}} \in \overline{\mathcal{T}}^s$. For $j = 1, \dots, \lambda$, since \mathcal{Y}^ℓ 's are \mathbf{F}^t -progressively measurable processes and since ν_j is a $\overline{\mathcal{T}}_s^t$ -stopping time, we see that Ξ_j is an $\overline{\mathcal{F}}_T^t$ -measurable random variable.

Let $j = 1, \dots, \lambda$. By (7.50),

$$\mathbb{E}_{\widehat{\mathbb{P}}}[\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \gamma_j)] = \mathbb{E}_{\widehat{\mathbb{P}}}[\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \gamma_j)] = \mathbb{E}_{\widehat{\mathbb{P}}}[\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \gamma_j')] = \mathbb{E}_t[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} R^{t,\omega}(\tau, \gamma_j')(\widehat{\mathcal{X}})]. \quad (7.52)$$

Given $\widetilde{\omega} \in \Omega^t$, since $0 \leq \gamma_j'(\widetilde{\omega}) - \widehat{\gamma}_j(\widetilde{\omega}) < \mathfrak{a}$, (1.6) implies that

$$\begin{aligned} R^{t,\omega}(\tau, \gamma_j')(\widetilde{\omega}) - R^{t,\omega}(\tau, \widehat{\gamma}_j)(\widetilde{\omega}) &= \int_{\tau(\widetilde{\omega}) \wedge \widehat{\gamma}_j(\widetilde{\omega})}^{\tau(\widetilde{\omega}) \wedge \gamma_j'(\widetilde{\omega})} g_r^{t,\omega}(\widetilde{\omega}) dr + \mathbf{1}_{\{\widehat{\gamma}_j(\widetilde{\omega}) < \tau(\widetilde{\omega}) \leq \gamma_j'(\widetilde{\omega})\}} (L^{t,\omega}(\tau(\widetilde{\omega}), \widetilde{\omega}) - U^{t,\omega}(\widehat{\gamma}_j(\widetilde{\omega}), \widetilde{\omega})) \\ &\quad + \mathbf{1}_{\{\gamma_j'(\widetilde{\omega}) < \tau(\widetilde{\omega})\}} (U^{t,\omega}(\gamma_j'(\widetilde{\omega}), \widetilde{\omega}) - U^{t,\omega}(\widehat{\gamma}_j(\widetilde{\omega}), \widetilde{\omega})) \\ &\leq \xi_{\mathfrak{m}}(\widetilde{\omega}) + \mathbf{1}_{\{\widehat{\gamma}_j(\widetilde{\omega}) < \tau(\widetilde{\omega}) \leq \gamma_j'(\widetilde{\omega})\}} \varrho_0 \left((\tau(\widetilde{\omega}) - \widehat{\gamma}_j(\widetilde{\omega})) + \sup_{r \in [0, T]} |(\omega \otimes_t \widetilde{\omega})(r \wedge \tau(\widetilde{\omega})) - (\omega \otimes_t \widetilde{\omega})(r \wedge \widehat{\gamma}_j(\widetilde{\omega}))| \right) \\ &\quad + \mathbf{1}_{\{\gamma_j'(\widetilde{\omega}) < \tau(\widetilde{\omega})\}} \varrho_0 \left((\gamma_j'(\widetilde{\omega}) - \widehat{\gamma}_j(\widetilde{\omega})) + \sup_{r \in [0, T]} |(\omega \otimes_t \widetilde{\omega})(r \wedge \gamma_j'(\widetilde{\omega})) - (\omega \otimes_t \widetilde{\omega})(r \wedge \widehat{\gamma}_j(\widetilde{\omega}))| \right) \\ &\leq \xi_{\mathfrak{m}}(\widetilde{\omega}) + \mathbf{1}_{\{\widehat{\gamma}_j(\widetilde{\omega}) < \tau(\widetilde{\omega}) \leq \gamma_j'(\widetilde{\omega})\}} \varrho_0 \left(\mathfrak{a} + \sup_{r \in [\widehat{\gamma}_j(\widetilde{\omega}), \tau(\widetilde{\omega})]} |\widetilde{\omega}(r) - \widetilde{\omega}(\widehat{\gamma}_j(\widetilde{\omega}))| \right) + \mathbf{1}_{\{\gamma_j'(\widetilde{\omega}) < \tau(\widetilde{\omega})\}} \varrho_0 \left(\mathfrak{a} + \sup_{r \in [\gamma_j'(\widetilde{\omega}), \gamma_j'(\widetilde{\omega})]} |\widetilde{\omega}(r) - \widetilde{\omega}(\widehat{\gamma}_j(\widetilde{\omega}))| \right) \\ &\leq \xi_{\mathfrak{m}}(\widetilde{\omega}) + \varrho_0 \left(\mathfrak{a} + \sup_{r \in [\nu_j(\widehat{\mathcal{W}}(\widetilde{\omega})), (\nu_j(\widehat{\mathcal{W}}(\widetilde{\omega})) + \mathfrak{a}) \wedge T]} |\widetilde{\omega}(r) - \widetilde{\omega}(\nu_j(\widehat{\mathcal{W}}(\widetilde{\omega})))| \right). \end{aligned}$$

Taking $\widetilde{\omega} = \widehat{\mathcal{X}}(\widetilde{\omega}')$, one can deduce from (7.44) that for \mathbb{P}_0^t -a.s. $\widetilde{\omega}' \in \Omega^t$,

$$R^{t,\omega}(\tau, \gamma_j')(\widehat{\mathcal{X}}(\widetilde{\omega}')) - R^{t,\omega}(\tau, \widehat{\gamma}_j)(\widehat{\mathcal{X}}(\widetilde{\omega}')) \leq \xi_{\mathfrak{m}}(\widehat{\mathcal{X}}(\widetilde{\omega}')) + \varrho_0 \left(\mathfrak{a} + \sup_{r \in [\nu_j(\widetilde{\omega}'), (\nu_j(\widetilde{\omega}') + \mathfrak{a}) \wedge T]} |\widehat{\mathcal{X}}_r(\widetilde{\omega}') - \widehat{\mathcal{X}}_{\nu_j}(\widetilde{\omega}')| \right). \quad (7.53)$$

Also, (7.44) and (7.47) show that for any $\widetilde{\omega}' \in (\mathcal{N}_R \cup \widehat{\mathcal{N}})^c$

$$\begin{aligned} R^{t,\omega}(\tau, \widehat{\gamma}_j)(\widehat{\mathcal{X}}(\widetilde{\omega}')) &= \int_t^{\widehat{\tau}(\widetilde{\omega}') \wedge \nu_j(\widetilde{\omega}')} \mathcal{Y}_r^1(\widehat{\mathcal{X}}(\widetilde{\omega}')) dr + \mathbf{1}_{\{\widehat{\tau}(\widetilde{\omega}') \leq \nu_j(\widetilde{\omega}')\}} \mathcal{Y}^2(\widehat{\tau}(\widetilde{\omega}'), \widehat{\mathcal{X}}(\widetilde{\omega}')) + \mathbf{1}_{\{\nu_j(\widetilde{\omega}') < \widehat{\tau}(\widetilde{\omega}')\}} \mathcal{Y}^3(\nu_j(\widetilde{\omega}'), \widehat{\mathcal{X}}(\widetilde{\omega}')) \\ &= \int_t^{\widehat{\tau}(\widetilde{\omega}') \wedge \nu_j(\widetilde{\omega}')} \mathcal{Y}_r^1(\widetilde{\omega}') dr + \mathbf{1}_{\{\widehat{\tau}(\widetilde{\omega}') \leq \nu_j(\widetilde{\omega}')\}} \mathcal{Y}^2(\widehat{\tau}(\widetilde{\omega}'), \widetilde{\omega}') + \mathbf{1}_{\{\nu_j(\widetilde{\omega}') < \widehat{\tau}(\widetilde{\omega}')\}} \mathcal{Y}^3(\nu_j(\widetilde{\omega}'), \widetilde{\omega}') = \Xi_j(\widetilde{\omega}'). \end{aligned} \quad (7.54)$$

Since $\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j) \in \overline{\mathcal{F}}_s^t$, $j = 0, \dots, \lambda$ by (5.4) and since ν_j 's are $\overline{\mathcal{T}}_s^t$ -stopping times, $\overline{\nu} := \mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A_0)} T + \sum_{j=1}^{\lambda} \mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A_j)} \nu_j$ is also a $\overline{\mathcal{T}}_s^t$ -stopping time. Set $\overline{\eta} := \sup_{r \in [\overline{\nu}, (\overline{\nu} + \mathfrak{a}) \wedge T]} |\widehat{\mathcal{X}}_r - \widehat{\mathcal{X}}_{\overline{\nu}}|$. Using the inequality $(a + b)^{\varpi} \leq 2^{\varpi-1}(a^{\varpi} + b^{\varpi})$, $\forall a, b > 0$, one can deduce from (7.54), (7.53) and (5.3) that

$$\begin{aligned} \sum_{j=1}^{\lambda} \mathbb{E}_t \left[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \left(R^{t,\omega}(\tau, \gamma_j')(\widehat{\mathcal{X}}) - \Xi_j \right) \right] &\leq \sum_{j=1}^{\lambda} \mathbb{E}_t \left[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \left(\xi_{\mathfrak{m}}(\widehat{\mathcal{X}}) + \varrho_0 \left(\mathfrak{a} + \sup_{r \in [\nu_j, (\nu_j + \mathfrak{a}) \wedge T]} |\widehat{\mathcal{X}}_r - \widehat{\mathcal{X}}_{\nu_j}| \right) \right) \right] \\ &= \sum_{j=1}^{\lambda} \mathbb{E}_t \left[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \left(\xi_{\mathfrak{m}}(\widehat{\mathcal{X}}) + \varrho_0(\mathfrak{a} + \overline{\eta}) \right) \right] \leq \mathbb{E}_t \left[\xi_{\mathfrak{m}}(\widehat{\mathcal{X}}) + \varrho_0(\mathfrak{a} + \overline{\eta}) \right] \\ &\leq \mathbb{E}_{\widehat{\mathbb{P}}}[\xi_{\mathfrak{m}}] + \mathbb{E}_t \left[\mathbf{1}_{\{\overline{\eta} \leq \mathfrak{a}^{\frac{1}{4}}\}} \varrho_0(\mathfrak{a} + \mathfrak{a}^{\frac{1}{4}}) + \kappa \mathbf{1}_{\{\overline{\eta} > \mathfrak{a}^{\frac{1}{4}}\}} (1 + (\mathfrak{a} + \overline{\eta})^{\varpi}) \right] \leq \mathbb{E}_{\widehat{\mathbb{P}}}[\xi_{\mathfrak{m}}] + \varrho_0(\mathfrak{a} + \mathfrak{a}^{\frac{1}{4}}) + \kappa \mathfrak{a}^{-1/4} \mathbb{E}_t \left[(1 + 2^{\varpi-1} \mathfrak{a}^{\varpi}) \overline{\eta} + 2^{\varpi-1} \overline{\eta}^{\varpi+1} \right] \\ &\leq \delta/2 + \varrho_0(\mathfrak{a} + \mathfrak{a}^{\frac{1}{4}}) + \kappa(1 + 2^{\varpi-1} \mathfrak{a}^{\varpi}) \varphi_1(\|\omega\|_{0,t}) \mathfrak{a}^{\frac{1}{4}} + \kappa 2^{\varpi-1} \varphi_{\varpi+1}(\|\omega\|_{0,t}) \mathfrak{a}^{\varpi/2+1/4} = \delta/2 + \Phi(\|\omega\|_{0,t}, 2^{-\mathfrak{m}}) \leq \delta. \end{aligned} \quad (7.55)$$

Then we see from (7.52) that

$$\sum_{j=1}^{\lambda} \mathbb{E}_{\widehat{\mathbb{P}}}[\mathbf{1}_{A \cap \mathcal{A}_j} R^{t,\omega}(\tau, \gamma_j)] = \sum_{j=1}^{\lambda} \mathbb{E}_t \left[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} R^{t,\omega}(\tau, \gamma_j')(\widehat{\mathcal{X}}) \right] \leq \sum_{j=1}^{\lambda} \mathbb{E}_t \left[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \Xi_j \right] + \delta, \quad \text{proving (7.51).}$$

2d) We are ready to use (2.1) and the estimate (5.2) to verify (2.8) for $\widehat{\mathbb{P}}$.

Let $j = 1, \dots, \lambda$ again. As $\widehat{\mathbb{P}} \in \widehat{\mathfrak{P}}_t$ by (7.43), (7.54), (2.5) and (2.6) imply that

$$\mathbb{E}_t[|\Xi_j|] \leq \mathbb{E}_t \left[\int_t^T |g_r^{t,\omega}(\widehat{\mathcal{X}})| dr + \Psi_*^{t,\omega}(\widehat{\mathcal{X}}) \right] = \mathbb{E}_{\widehat{\mathbb{P}}} \left[\int_t^T |g_r^{t,\omega}| dr + \Psi_*^{t,\omega} \right] = \mathbb{E}_{\widehat{\mathbb{P}}} \left[\int_t^T |g_r^{t,\omega}| dr + \Psi_*^{t,\omega} \right] < \infty.$$

Since $\hat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j) \in \bar{\mathcal{F}}_s^t$, applying Lemma A.2 (1) of [5] with $(\mathbb{P}, X, \xi) = (\mathbb{P}_0^t, B^t, \Xi_j)$, using (u4) with $A = A \cap \mathcal{A}_j$ and applying Proposition 2.3 in the ArXiv version of [5] with $(\mathbb{P}, \xi) = (\mathbb{P}_0^t, \Xi_j)$, we can deduce from Proposition 1.2 (1) and (u1) that

$$\begin{aligned} \mathbb{E}_t[\mathbf{1}_{\hat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \Xi_j] &= \mathbb{E}_t[\mathbf{1}_{\hat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \mathbb{E}_t[\Xi_j | \bar{\mathcal{F}}_s^t]] = \mathbb{E}_t[\mathbf{1}_{\hat{\mathcal{X}}^{-1}(A \cap \mathcal{A}_j)} \mathbb{E}_t[\Xi_j | \mathcal{F}_s^t]] = \mathbb{E}_t[\mathbf{1}_{\mathcal{X}^{-1}(A \cap \mathcal{A}_j)} \mathbb{E}_t[\Xi_j | \mathcal{F}_s^t]] \\ &= \mathbb{E}_t[\mathbf{1}_{\{\tilde{\omega} \in \mathcal{X}^{-1}(A) \cap \mathcal{A}_j^{\mathcal{X}}\}} \mathbb{E}_s[\Xi_j^{s, \tilde{\omega}}]] = \mathbb{E}_t[\mathbf{1}_{\{\tilde{\omega} \in \mathcal{X}^{-1}(A) \cap \mathcal{A}_j^{\mathcal{X}} \cap \tilde{\mathcal{A}}_j\}} \mathbb{E}_s[\Xi_j^{s, \tilde{\omega}}]] \end{aligned} \quad (7.56)$$

Let $\tilde{\omega} \in \mathcal{A}_j^{\mathcal{X}} \cap \tilde{\mathcal{A}}_j \cap \tilde{\mathcal{N}}_j^c \cap \tilde{\mathcal{N}}_R^c \cap \mathcal{N}_\tau^c$. As $\hat{\tau}^{s, \tilde{\omega}} \in \bar{\mathcal{T}}^s$, similar to $\hat{\gamma}_j = \nu_j(\hat{\mathcal{W}})$, $\zeta_{\tilde{\omega}} := \hat{\tau}^{s, \tilde{\omega}}(\mathcal{W}^j)$ is a \mathfrak{F}^j -stopping time. Let $\tilde{\omega} \in \Omega^s$ such that $\tilde{\omega}$ is not in the \mathbb{P}_0^s -null set $(\mathcal{N}_R \cup \hat{\mathcal{N}})^{s, \tilde{\omega}} \cup \mathcal{N}_{\mathcal{X}^j} \cup \mathcal{N}_{\tilde{\omega}}$, and define $\Delta X_{\tilde{\omega}}^j(\tilde{\omega}) := \|X^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega}), \mu^j}(\tilde{\omega}) - \mathcal{X}^j(\tilde{\omega})\|_{s, T}$. Taking $\tilde{\omega}' = \tilde{\omega} \otimes_s \tilde{\omega} \in (\mathcal{N}_R \cup \hat{\mathcal{N}})^c$ in (7.54), we see from (2.3), (7.44), (u3), (2.1) as well as an analogy to the second equality of (7.4) that

$$\begin{aligned} \Xi_j^{s, \tilde{\omega}}(\tilde{\omega}) &= R^{t, \omega}(\tau, \hat{\gamma}_j)(\hat{\mathcal{X}}(\tilde{\omega} \otimes_s \tilde{\omega})) = R(t, \tau(\hat{\mathcal{X}}(\tilde{\omega} \otimes_s \tilde{\omega})), \hat{\gamma}_j(\hat{\mathcal{X}}(\tilde{\omega} \otimes_s \tilde{\omega})), \omega \otimes_t (\hat{\mathcal{X}}(\tilde{\omega} \otimes_s \tilde{\omega}))) \\ &= R(t, \hat{\tau}(\tilde{\omega} \otimes_s \tilde{\omega}), \nu_j(\tilde{\omega} \otimes_s \tilde{\omega}), \omega \otimes_t (\hat{\mathcal{X}}(\tilde{\omega} \otimes_s \tilde{\omega}))) = R(t, \hat{\tau}^{s, \tilde{\omega}}(\tilde{\omega}), \wp_j(\tilde{\omega}), \omega \otimes_t (\hat{\mathcal{X}}(\tilde{\omega} \otimes_s \tilde{\omega}))) \\ &= R(t, \zeta_{\tilde{\omega}}(\mathcal{X}^j(\tilde{\omega})), \wp(\mathcal{X}^j(\tilde{\omega})), \omega \otimes_t (\mathcal{X}(\tilde{\omega}) \otimes_s X^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega}), \mu^j}(\tilde{\omega}))) \\ &\leq R(t, \zeta_{\tilde{\omega}}(\mathcal{X}^j(\tilde{\omega})), \wp(\mathcal{X}^j(\tilde{\omega})), (\omega \otimes_t \mathcal{X}(\tilde{\omega})) \otimes_s (\mathcal{X}^j(\tilde{\omega}))) + (1+T)\varrho_0(\Delta X_{\tilde{\omega}}^j(\tilde{\omega})) \\ &= R(s, \zeta_{\tilde{\omega}}(\mathcal{X}^j(\tilde{\omega})), \wp(\mathcal{X}^j(\tilde{\omega})), (\omega \otimes_t \mathcal{X}(\tilde{\omega})) \otimes_s (\mathcal{X}^j(\tilde{\omega}))) + \int_t^s g_r((\omega \otimes_t \mathcal{X}(\tilde{\omega})) \otimes_s (\mathcal{X}^j(\tilde{\omega}))) dr + (1+T)\varrho_0(\Delta X_{\tilde{\omega}}^j(\tilde{\omega})) \\ &= (R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}, \wp))(\mathcal{X}^j(\tilde{\omega})) + \int_t^s g_r(\omega \otimes_t \mathcal{X}(\tilde{\omega})) dr + (1+T)\varrho_0(\Delta X_{\tilde{\omega}}^j(\tilde{\omega})). \end{aligned}$$

Since $\varrho_0(\Delta X_{\tilde{\omega}}^j(\tilde{\omega})) \leq \mathbf{1}_{\{\Delta X_{\tilde{\omega}}^j(\tilde{\omega}) \leq \delta^{1/2}\}} \varrho_0(\delta^{1/2}) + \mathbf{1}_{\{\Delta X_{\tilde{\omega}}^j(\tilde{\omega}) > \delta^{1/2}\}} \kappa \delta^{-1/2} (\Delta X_{\tilde{\omega}}^j(\tilde{\omega}) + (\Delta X_{\tilde{\omega}}^j(\tilde{\omega}))^{\varpi+1})$, (5.2) shows that

$$\begin{aligned} \mathbb{E}_s[\Xi_j^{s, \tilde{\omega}}] &\leq \mathbb{E}_s[(R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}, \wp))(\mathcal{X}^j)] + \int_t^s g_r^{t, \omega}(\mathcal{X}(\tilde{\omega})) dr + (1+T)\varrho_0(\delta^{1/2}) \\ &\quad + (1+T)\kappa \delta^{-1/2} (C_1 T \|\omega \otimes_t \mathcal{X}(\tilde{\omega}) - \omega \otimes_t \tilde{\omega}_j\|_{0, s} + C_{\varpi+1} T^{\varpi+1} \|\omega \otimes_t \mathcal{X}(\tilde{\omega}) - \omega \otimes_t \tilde{\omega}_j\|_{0, s}^{\varpi+1}). \end{aligned} \quad (7.57)$$

Set $\hat{\varrho}_0(\delta) := \delta + (1+T)\varrho_0(\delta^{1/2}) + (1+T)\kappa(C_1 T \delta^{1/2} + C_{\varpi+1} T^{\varpi+1} \delta^{\varpi+1/2})$. As $\tilde{\omega} \in \mathcal{A}_j^{\mathcal{X}} = \mathcal{X}^{-1}(\mathcal{A}_j)$, i.e. $\mathcal{X}(\tilde{\omega}) \in \mathcal{A}_j \subset O_{\delta_j}^s(\tilde{\omega}_j)$, one has $\|\omega \otimes_t \mathcal{X}(\tilde{\omega}) - \omega \otimes_t \tilde{\omega}_j\|_{0, s} = \|\mathcal{X}(\tilde{\omega}) - \tilde{\omega}_j\|_{t, s} < \delta_j \leq \delta$. It follows from (7.57) that

$$\begin{aligned} \mathbb{E}_s[\Xi_j^{s, \tilde{\omega}}] &\leq \mathbb{E}_{\mathbb{P}^j}[R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}, \wp)] + \int_t^s g_r^{t, \omega}(\mathcal{X}(\tilde{\omega})) dr + \hat{\varrho}_0(\delta) - \delta \\ &\leq \sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}^j}[R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\mathcal{X}(\tilde{\omega})) dr + \hat{\varrho}_0(\delta) - \delta. \end{aligned} \quad (7.58^*)$$

Plugging this back into (7.56), we see from (7.51) and (u1) that

$$\begin{aligned} \sum_{j=1}^{\lambda} \mathbb{E}_{\hat{\mathbb{P}}}[\mathbf{1}_{A \cap \mathcal{A}_j} R^{t, \omega}(\tau, \gamma_j)] &\leq \sum_{j=1}^{\lambda} \mathbb{E}_t[\mathbf{1}_{\{\tilde{\omega} \in \mathcal{X}^{-1}(A) \cap \mathcal{X}^{-1}(\mathcal{A}_j)\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}^j}[R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\mathcal{X}(\tilde{\omega})) dr + \hat{\varrho}_0(\delta) - \delta \right)] + \delta \\ &= \sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}^j}[R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr + \hat{\varrho}_0(\delta) - \delta \right)] + \delta \\ &= \sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\tilde{\omega} \in A \cap \mathcal{A}_j\}} \left(\sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}^j}[R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)] + \int_t^s g_r^{t, \omega}(\tilde{\omega}) dr \right)] + \mathbb{P}(A \cap \mathcal{A}_0^c)(\hat{\varrho}_0(\delta) - \delta) + \delta. \end{aligned}$$

In the last equality, we used the fact that the mapping $\tilde{\omega} \rightarrow \sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbb{P}^j}[R^{s, \omega \otimes_t \tilde{\omega}}(\varsigma, \wp)]$ is continuous under norm

$\|\cdot\|_{t, T}$ and thus \mathcal{F}_T^t -measurable by Remark 2.2 (2). Therefore, (2.8) holds for $\wp_j^n = \gamma_j$, $j=1, \dots, \lambda$.

3) In this part, we still use (2.1) and the estimate (5.2) to show that $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ satisfies Assumption 3.1.

Fix $n \in \mathbb{N} \cup \{\infty\}$, $t \in [0, T]$, $\omega, \omega' \in \Omega$, $\mu \in \mathcal{U}_t$ and set $\delta := \|\omega' - \omega\|_{0,t}$. We still take the notation (7.46) and set $(\mathbb{P}', \mathbf{p}', \mathcal{X}', \mathcal{W}', \mathfrak{F}') := (\mathbb{P}^{t, \omega', \mu}, \mathbf{p}^{t, \omega', \mu}, X^{t, \omega', \mu}, \widetilde{W}^{t, \omega', \mu}, \mathfrak{F}^{t, \omega', \mu})$.

Fix $\varepsilon > 0$. We still define ξ_m 's as in (7.48) and can find a $\mathfrak{k} \in \mathbb{N}$ such that $\mathbb{E}_{\mathbb{P}'}[\xi_{\mathfrak{k}}] \leq \varepsilon/2$ and $\Phi(\|\omega'\|_{0,t}, 2^{-\mathfrak{k}}) \leq \varepsilon/2$. Also, fix $\gamma \in \mathcal{T}^t$ and $\tau \in \mathcal{T}^t(n)$. Similar to $\hat{\tau} = \tau(\hat{\mathcal{X}})$ in part 2c), $\tau(\mathcal{X}')$ belongs to $\overline{\mathcal{T}}^t$; and analogous to $\hat{\gamma}_j = \nu_j(\hat{\mathcal{W}})$, (7.45) implies that $\tilde{\tau} := \tau(\mathcal{X}'(\mathcal{W}))$ is a \mathfrak{F} -stopping time. Symmetrically, $\gamma(\mathcal{X})$ belongs to $\overline{\mathcal{T}}^t$ and $\tilde{\gamma} := \gamma(\mathcal{X}(\mathcal{W}'))$ defines a \mathfrak{F}' -stopping time.

Set $t_i := t \vee (i2^{-\mathfrak{k}}T)$, $i = 0, \dots, 2^{\mathfrak{k}}$. Then $\tilde{\gamma}'_{\mathfrak{k}} := \sum_{i=0}^{2^{\mathfrak{k}}} \mathbf{1}_{\{t_{i-1} < \tilde{\gamma} \leq t_i\}} t_i$ defines a \mathfrak{F}' -stopping time, where $t_{-1} := -1$. By similar arguments to those that lead to (7.50), one can construct a \mathcal{T}^t -stopping time $\tilde{\gamma}_{\mathfrak{k}}$ valued in $\{t_i\}_{i=0}^{2^{\mathfrak{k}}}$ such that $\tilde{\gamma}'_{\mathfrak{k}} = \tilde{\gamma}_{\mathfrak{k}}$, \mathbf{p}' -a.s. Analogous to (7.53), we can deduce that for \mathbb{P}'_0 -a.s. $\tilde{\omega} \in \Omega^t$,

$$R^{t, \omega}(\tau, \tilde{\gamma}'_{\mathfrak{k}})(\mathcal{X}'(\tilde{\omega})) - R^{t, \omega}(\tau, \tilde{\gamma})(\mathcal{X}'(\tilde{\omega})) \leq \xi_{\mathfrak{k}}(\mathcal{X}'(\tilde{\omega})) + \varrho_0(2^{-\mathfrak{k}} + \eta'(\tilde{\omega})),$$

where $\eta' := \sup_{r \in [\gamma(\mathcal{X}), (\gamma(\mathcal{X}) + 2^{-\mathfrak{k}}) \wedge T]} |\mathcal{X}'_r - \mathcal{X}'_{\gamma(\mathcal{X})}|$. And similar to (7.55), (5.3) implies that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}'}[R^{t, \omega}(\tau, \tilde{\gamma}'_{\mathfrak{k}}) - R^{t, \omega}(\tau, \tilde{\gamma})] &= \mathbb{E}_{\mathbb{P}'}[R^{t, \omega}(\tau, \tilde{\gamma}'_{\mathfrak{k}}) - R^{t, \omega}(\tau, \tilde{\gamma})] = \mathbb{E}_t[R^{t, \omega}(\tau, \tilde{\gamma}'_{\mathfrak{k}})(\mathcal{X}') - R^{t, \omega}(\tau, \tilde{\gamma})(\mathcal{X}')] \\ &\leq \mathbb{E}_t[\xi_{\mathfrak{k}}(\mathcal{X}') + \varrho_0(2^{-\mathfrak{k}} + \eta')] \leq \mathbb{E}_{\mathbb{P}'}[\xi_{\mathfrak{k}}] + \Phi(\|\omega'\|_{0,t}, 2^{-\mathfrak{k}}) \leq \varepsilon. \end{aligned} \quad (7.59)$$

Since (7.44) shows that $\tau(\mathcal{X}'(\tilde{\omega})) = \tau(\mathcal{X}'(\mathcal{W}(\mathcal{X}'(\tilde{\omega})))) = \tilde{\tau}(\mathcal{X}'(\tilde{\omega}))$ and $\tilde{\gamma}(\mathcal{X}'(\tilde{\omega})) = \gamma(\mathcal{X}(\mathcal{W}'(\mathcal{X}'(\tilde{\omega})))) = \gamma(\mathcal{X}(\tilde{\omega}))$ hold for \mathbb{P}'_0 -a.s. $\tilde{\omega} \in \Omega^t$, we see from (2.3) and (2.1) that for \mathbb{P}'_0 -a.s. $\tilde{\omega} \in \Omega^t$

$$\begin{aligned} (R^{t, \omega'}(\tau, \tilde{\gamma}))(\mathcal{X}'(\tilde{\omega})) - (R^{t, \omega}(\tilde{\tau}, \gamma))(\mathcal{X}'(\tilde{\omega})) &= R(t, \tau(\mathcal{X}'(\tilde{\omega})), \tilde{\gamma}(\mathcal{X}'(\tilde{\omega})), \omega' \otimes_t \mathcal{X}'(\tilde{\omega})) - R(t, \tilde{\tau}(\mathcal{X}'(\tilde{\omega})), \gamma(\mathcal{X}'(\tilde{\omega})), \omega \otimes_t \mathcal{X}'(\tilde{\omega})) \\ &= R(t, \tilde{\tau}(\mathcal{X}'(\tilde{\omega})), \gamma(\mathcal{X}'(\tilde{\omega})), \omega' \otimes_t \mathcal{X}'(\tilde{\omega})) - R(t, \tilde{\tau}(\mathcal{X}'(\tilde{\omega})), \gamma(\mathcal{X}'(\tilde{\omega})), \omega \otimes_t \mathcal{X}'(\tilde{\omega})) \\ &\leq (1+T)\varrho_0(\|\omega' \otimes_t \mathcal{X}'(\tilde{\omega}) - \omega \otimes_t \mathcal{X}'(\tilde{\omega})\|_{0,T}) \leq (1+T)\varrho_0(\|\omega' - \omega\|_{0,t} + \|\mathcal{X}'(\tilde{\omega}) - \mathcal{X}(\tilde{\omega})\|_{t,T}) = (1+T)\varrho_0(\delta + \Delta X(\tilde{\omega})) \\ &\leq \mathbf{1}_{\{\Delta X(\tilde{\omega}) \leq \delta^{1/2}\}}(1+T)\varrho_0(\delta + \delta^{1/2}) + \mathbf{1}_{\{\Delta X(\tilde{\omega}) > \delta^{1/2}\}}\kappa(1+T)\delta^{-1/2}((1+2^{\varpi-1}\delta^{\varpi})\Delta X(\tilde{\omega}) + 2^{\varpi-1}(\Delta X(\tilde{\omega}))^{\varpi+1}), \end{aligned}$$

with $\Delta X(\tilde{\omega}) := \|\mathcal{X}'(\tilde{\omega}) - \mathcal{X}(\tilde{\omega})\|_{t,T}$. Then (7.59) and (5.2) show that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}'}[R^{t, \omega}(\tau, \tilde{\gamma}_{\mathfrak{k}})] &= \mathbb{E}_{\mathbb{P}'}[R^{t, \omega}(\tau, \tilde{\gamma}_{\mathfrak{k}})] \leq \mathbb{E}_{\mathbb{P}'}[R^{t, \omega'}(\tau, \tilde{\gamma})] + \varepsilon = \mathbb{E}_t[(R^{t, \omega'}(\tau, \tilde{\gamma}))(\mathcal{X}')] + \varepsilon \\ &\leq \mathbb{E}_t[(R^{t, \omega}(\tilde{\tau}, \gamma))(\mathcal{X}')] + \varrho_1(\delta) + \varepsilon = \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tilde{\tau}, \gamma)] + \varrho_1(\delta) + \varepsilon, \end{aligned} \quad (7.60)$$

where $\varrho_1(\delta) := (1+T)\varrho_0(\delta + \delta^{1/2}) + \kappa(1+T)((1+2^{\varpi-1}\delta^{\varpi})C_1T\delta^{1/2} + 2^{\varpi-1}C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2}) \geq \varrho_0(\delta)$.

Similar to (7.58), one can deduce that $\mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tilde{\tau}, \gamma)] \leq \sup_{\varsigma \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\varsigma, \gamma)]$. So it follows from (7.60) that

$$\mathbb{E}_{\mathbb{P}'}[R^{t, \omega'}(\tau, \tilde{\gamma}_{\mathfrak{k}})] \leq \sup_{\varsigma \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\varsigma, \gamma)] + \varrho_1(\delta) + \varepsilon.$$

Taking supremum over $\tau \in \mathcal{T}^t(n)$ on the left-hand-side yields that

$$\inf_{\zeta \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}'}[R^{t, \omega'}(\tau, \zeta)] \leq \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}'}[R^{t, \omega'}(\tau, \tilde{\gamma}_{\mathfrak{k}})] \leq \sup_{\varsigma \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\varsigma, \gamma)] + \varrho_1(\delta) + \varepsilon.$$

Then taking infimum over $\gamma \in \mathcal{T}^t$ on the right-hand-side, we obtain that

$$\inf_{\zeta \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}^{t, \omega', \mu}}[R^{t, \omega'}(\tau, \zeta)] \leq \inf_{\gamma \in \mathcal{T}^t} \sup_{\varsigma \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}^{t, \omega, \mu}}[R^{t, \omega}(\varsigma, \gamma)] + \varrho_1(\delta) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ and taking infimum over $\mu \in \mathcal{U}_t$ on both sides lead to that

$$V_t^n(\omega') = \inf_{\mu \in \mathcal{U}_t} \inf_{\zeta \in \mathcal{T}^t} \sup_{\tau \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}^{t, \omega', \mu}}[R^{t, \omega'}(\tau, \zeta)] \leq \inf_{\mu \in \mathcal{U}_t} \inf_{\gamma \in \mathcal{T}^t} \sup_{\varsigma \in \mathcal{T}^t(n)} \mathbb{E}_{\mathbb{P}^{t, \omega, \mu}}[R^{t, \omega}(\varsigma, \gamma)] + \varrho_1(\|\omega' - \omega\|_{0,t}) = V_t^n(\omega) + \varrho_1(\|\omega' - \omega\|_{0,t}).$$

Exchanging the roles of ω' and ω shows that $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$ satisfies (3.4).

4) To verify Assumption 3.2 for $\{\mathcal{P}(t, \omega)\}_{(t, \omega) \in [0, T] \times \Omega}$, we fix $\alpha > 0$ and $\delta \in (0, T]$.

Let $t \in [0, T]$, $\omega \in O_\alpha^t(\mathbf{0})$, $\mu \in \mathcal{U}_t$ and $\zeta \in \mathcal{T}^t$. We take the notation (7.46) again. Similar to $\hat{\tau} = \tau(\hat{\mathcal{X}})$ in part 2c), $\tilde{\zeta} := \zeta(\mathcal{X})$ is a $\overline{\mathcal{T}}^t$ -stopping time. Set $\tilde{\eta} := \sup_{r \in [\tilde{\zeta}, (\tilde{\zeta} + \delta) \wedge T]} |\mathcal{X}_r - \mathcal{X}_{\tilde{\zeta}}|$. Analogous to (7.55), one can deduce from (5.3) that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\varrho_1 \left(\delta + \sup_{r \in [\zeta, (\zeta + \delta) \wedge T]} |B_r^t - B_\zeta^t| \right) \right] &= \mathbb{E}_{\mathbb{P}} \left[\varrho_1 \left(\delta + \sup_{r \in [\zeta, (\zeta + \delta) \wedge T]} |B_r^t - B_\zeta^t| \right) \right] = \mathbb{E}_t \left[\varrho_1 \left(\delta + \sup_{r \in [\tilde{\zeta}, (\tilde{\zeta} + \delta) \wedge T]} |\mathcal{X}_r - \mathcal{X}_{\tilde{\zeta}}| \right) \right] = \mathbb{E}_t [\varrho_1(\delta + \tilde{\eta})] \\ &\leq \varrho_1(\delta + \delta^{1/4}) + \kappa(1 + 2^{\varpi-1} \delta^{\varpi}) \varphi_1(\|\omega\|_{0,t}) \delta^{1/4} + \kappa 2^{\varpi-1} \varphi_{\varpi+1}(\|\omega\|_{0,t}) \delta^{\varpi/2+1/4} \leq \varrho_\alpha(\delta), \end{aligned}$$

where $\varrho_\alpha(\delta) := \varrho_1(\delta + \delta^{1/4}) + \kappa(1 + 2^{\varpi-1} \delta^{\varpi}) \varphi_1(\alpha) \delta^{1/4} + \kappa 2^{\varpi-1} \varphi_{\varpi+1}(\alpha) \delta^{\varpi/2+1/4}$. Taking supremum over $\zeta \in \mathcal{T}^t$ and then taking supremum over $\mu \in \mathcal{U}_t$ and $\omega \in O_\alpha^t(\mathbf{0})$ yield (3.6). \square

7.5 Proof of Theorem 6.1

If $V_0 = L_0$, then $\tau_* = 0$ and it thus holds for any $(\mathbb{P}, \gamma) \in \mathcal{P} \times \mathcal{T}$ that $\mathbb{E}_{\mathbb{P}}[R(\tau_*, \gamma)] = \mathbb{E}_{\mathbb{P}}[R(0, \gamma)] = \mathbb{E}_{\mathbb{P}}[L_0] = L_0 = V_0$.

Next, let us assume that $V_0 > L_0$. Theorem 4.1 (1), Proposition 3.4 (1), (A') and the proof of Remark 3.1 imply that the process $\mathcal{X}_t := V_t - L_t$, $t \in [0, T]$ has all continuous paths and satisfies

$$|\mathcal{X}_t(\omega) - \mathcal{X}_t(\omega')| \leq |V_t(\omega) - V_t(\omega')| + |L_t(\omega) - L_t(\omega')| \leq 2\rho_0(\|\omega - \omega'\|_{0,t}), \quad \forall t \in [0, T], \quad \forall \omega, \omega' \in \Omega.$$

Then applying Theorem 3.1 of [7] with payoff processes $\mathcal{L} := -U$, $\mathcal{U} := -L$ and random maturity $\tau_0 = \inf\{t \in [0, T] : \mathcal{X}_t \leq 0\} \wedge T = \inf\{t \in [0, T] : V_t = L_t\} = \tau_*$ shows that (In particular, (H4) implies (P4) of [7] by Remark 3.1 (3) therein) for some $(\mathbb{P}_*, \gamma_*) \in \mathcal{P} \times \mathcal{T}$, $\sup_{(\mathbb{P}, \gamma) \in \mathcal{P} \times \mathcal{T}} \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\gamma < \tau_*\}} \mathcal{L}_\gamma + \mathbf{1}_{\{\tau_* \leq \gamma\}} \mathcal{U}_{\tau_*}] = \mathbb{E}_{\mathbb{P}_*}[\mathbf{1}_{\{\gamma_* < \tau_*\}} \mathcal{L}_{\gamma_*} + \mathbf{1}_{\{\tau_* \leq \gamma_*\}} \mathcal{U}_{\tau_*}]$. Multiplying -1 on both sides, we see from (4.3) that $V_0 = \inf_{(\mathbb{P}, \gamma) \in \mathcal{P} \times \mathcal{T}} \mathbb{E}_{\mathbb{P}}[R(\tau_*, \gamma)] = \mathbb{E}_{\mathbb{P}_*}[R(\tau_*, \gamma_*)]$. \square

A Appendix

A.1 A Technical Lemma

Lemma A.1. Define $\overline{\Upsilon}_t := \overline{V}_t + \int_0^t g_r dr$, $t \in [0, T]$ Given $\zeta \in \mathcal{T}$, it holds for any $(t, \omega) \in [0, T] \times \Omega$ and $t' \in [0, t]$ that

$$\overline{\Upsilon}_t(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[\left(\overline{\Upsilon}_{(\tau_{(t', \omega)}^*, (\Pi_{t'}^0) \wedge \zeta) \vee t} \right)^{t, \omega} \right]. \quad (\text{A.1})$$

Proof of Lemma A.1: Fix $0 \leq t' \leq t \leq T$, $\omega \in \Omega$, $\zeta \in \mathcal{T}$ and set $\alpha := 1 + \|\omega\|_{0,T}$.

- 1) When $t = T$, one has $\inf_{\mathbb{P} \in \mathcal{P}(T, \omega)} \mathbb{E}_{\mathbb{P}} \left[\left(\overline{\Upsilon}_{(\tau_{(t', \omega)}^*, (\Pi_{t'}^0) \wedge \zeta) \vee T} \right)^{T, \omega} \right] = \inf_{\mathbb{P} \in \mathcal{P}(T, \omega)} \mathbb{E}_{\mathbb{P}} \left[(\overline{\Upsilon}_T)^{T, \omega} \right] = \inf_{\mathbb{P} \in \mathcal{P}(T, \omega)} \mathbb{E}_{\mathbb{P}} [\overline{\Upsilon}_T(\omega)] = \overline{\Upsilon}_T(\omega)$.
- 2) Next, suppose that $t < T$ and $\overline{V}_t(\omega) = L_t(\omega)$. Then

$$\tau_{(t', \omega)}^*(\Pi_{t'}^0(\omega)) = \inf \left\{ s \in [t', T] : \overline{V}_s^{t', \omega}(\Pi_{t'}^0(\omega)) = L_s^{t', \omega}(\Pi_{t'}^0(\omega)) \right\} = \inf \left\{ s \in [t', T] : \overline{V}_s(\omega) = L_s(\omega) \right\} \leq t,$$

which means that $\omega \in (\Pi_{t'}^0)^{-1}(A')$ with $A' := \{\omega' \in \Omega^{t'} : \tau_{(t', \omega)}^*(\omega') \leq t\} \in \mathcal{F}_t^{t'}$. Since Lemma A.1 of [5] shows that

$$\Pi_{t'}^0 \text{ is an } \mathcal{F}_r / \mathcal{F}_r^{t'} \text{-measurable mapping,} \quad \forall r \in [t', T], \quad (\text{A.2})$$

we see that $(\Pi_{t'}^0)^{-1}(A') \in \mathcal{F}_t$. It follows from Lemma 1.1 that

$$\omega \otimes_t \Omega^t \subset (\Pi_{t'}^0)^{-1}(A') \quad \text{or} \quad \tau_{(t', \omega)}^*(\Pi_{t'}^0(\omega \otimes_t \tilde{\omega})) \leq t, \quad \forall \tilde{\omega} \in \Omega^t. \quad (\text{A.3})$$

Remark 3.2 and Proposition 3.4 (1) show that $\overline{\Upsilon}$ is an \mathbf{F} -adapted process with all continuous paths. Applying (1.8) to $\overline{\Upsilon}_t \in \mathcal{F}_t$ and using (A.3) yield that

$$\left(\overline{\Upsilon}_{(\tau_{(t', \omega)}^*, (\Pi_{t'}^0) \wedge \zeta) \vee t} \right)^{t, \omega}(\tilde{\omega}) = \overline{\Upsilon} \left((\tau_{(t', \omega)}^*(\Pi_{t'}^0(\omega \otimes_t \tilde{\omega})) \wedge \zeta(\omega \otimes_t \tilde{\omega})) \vee t, \omega \otimes_t \tilde{\omega} \right) = \overline{\Upsilon}_t(\omega \otimes_t \tilde{\omega}) = \overline{\Upsilon}_t(\omega), \quad \forall \tilde{\omega} \in \Omega^t,$$

Thus we still obtain (A.1) as an equality.

3) The discussion of the case $t < T$ with $\bar{V}_t(\omega) > L_t(\omega)$ is relatively lengthy. We split it into several steps. Since $\lim_{n \rightarrow \infty} \uparrow V_t^n(\omega) = \bar{V}_t(\omega)$ by (3.1) and Proposition 3.3, there exists an integer $N = N(t, \omega) > \log_2 \left(\frac{T}{T-t} \right)$ such that $V_t^n(\omega) > L_t(\omega)$ for any $n \geq N$.

Fix $\delta > 0$ and $k, n \in \mathbb{N}$ with $k \geq n > N$. For any $r \in [t', T]$, as $A_r := \{\tilde{\omega} \in \Omega^{t'} : \tau_{(t', \omega)}^{n, \delta}(\tilde{\omega}) < r\} \in \mathcal{F}_r^{t'}$, (A.2) implies that $\{\omega' \in \Omega : \tau_{(t', \omega)}^{n, \delta}(\Pi_{t'}^0(\omega')) < r\} = \{\omega' \in \Omega : \Pi_{t'}^0(\omega') \in A_r\} = (\Pi_{t'}^0)^{-1}(A_r) \in \mathcal{F}_r$. So $\tau_{(t', \omega)}^{n, \delta}(\Pi_{t'}^0)$ is an \mathbf{F} -optional time valued in $[t', T]$, and it follows that $\nu^{n, \delta} := (\tau_{(t', \omega)}^{n, \delta}(\Pi_{t'}^0) \wedge \zeta) \vee t$ is an \mathbf{F} -optional time valued in $[t, T]$.

Let i_k be the largest integer such that $i_k 2^{-k} T \leq t$. As $k > \log_2 \left(\frac{T}{T-t} \right)$, one can deduce that $i_k < 2^k - 1$. Set $t_{i_k} := t$ and $t_i := i 2^{-k} T$ for $i = i_k + 1, \dots, 2^k$.

3a) In the first step, we derive from Proposition 3.1 an auxiliary inequality:

$$V_{t_i}^n(\omega) \leq L_{t_i}(\omega) \vee \underline{\mathcal{E}}_{t_i} \left[V_{t_{i+1}}^n + \int_{t_i}^{t_{i+1}} g_r dr \right](\omega), \quad i = i_k, \dots, 2^k - 1. \quad (\text{A.4})$$

Let $i = i_k, \dots, 2^k - 1$. Applying (3.5) with $(t, s) = (t_i, t_{i+1})$ and taking $\gamma = t_{i+1}$ yield that

$$V_{t_i}^n(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t_i, \omega)} \sup_{\tau \in \mathcal{T}^{t_i}(n)} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau < t_{i+1}\}} R^{t_i, \omega}(\tau, t_{i+1}) + \mathbf{1}_{\{\tau \geq t_{i+1}\}} \left((V_{t_{i+1}}^n)^{t_i, \omega} + \int_{t_i}^{t_{i+1}} g_r^{t_i, \omega} dr \right) \right]. \quad (\text{A.5})$$

For any $\tau \in \mathcal{T}^{t_i}(n)$, it takes values in $\{t_i\} \cup \{j 2^{-n} T\}_{j=j_0}^{2^n}$, where j_0 is the smallest integer such that $t_i < j_0 2^{-n} T$. As $n \leq k$, one has $t_{i+1} \leq j_0 2^{-n} T$, so $\{\tau < t_{i+1}\} = \{\tau = t_i\} \in \mathcal{F}_{t_i}^{t_i} = \{\emptyset, \Omega^{t_i}\}$. To wit, we have either $\{\tau < t_{i+1}\} = \{\tau = t_i\} = \Omega^{t_i}$ or $\{\tau \geq t_{i+1}\} = \Omega^{t_i}$. Since $R^{t_i, \omega}(t_i, t_{i+1}) = L_{t_i}^{t_i, \omega} = L(t_i, \omega)$ by (7.6), we see from (A.5) that

$$V_{t_i}^n(\omega) \leq \inf_{\mathbb{P} \in \mathcal{P}(t_i, \omega)} \left(L_{t_i}(\omega) \vee \mathbb{E}_{\mathbb{P}} \left[(V_{t_{i+1}}^n)^{t_i, \omega} + \int_{t_i}^{t_{i+1}} g_r^{t_i, \omega} dr \right] \right) = L_{t_i}(\omega) \vee \underline{\mathcal{E}}_{t_i} \left[V_{t_{i+1}}^n + \int_{t_i}^{t_{i+1}} g_r dr \right](\omega), \quad \text{proving (A.4).}$$

3b) In the next step, we will show that over time grids $\{t_i\}_{i_k}^{2^k}$, the \mathbf{F} -adapted process $\Upsilon_t^n := V_t^n + \int_0^t g_r dr$, $t \in [0, T]$ is an $\underline{\mathcal{E}}$ -submartingale up to time $\nu_k^{n, \delta} := \sum_{i=i_k+1}^{2^k} \mathbf{1}_{\{t_{i-1} \leq \nu^{n, \delta} < t_i\}} t_i + \mathbf{1}_{\{\nu^{n, \delta} = T\}} T$, i.e.

$$\Upsilon_{\nu_k^{n, \delta} \wedge t_i}^n(\omega) \leq \underline{\mathcal{E}}_{t_i} \left[\Upsilon_{\nu_k^{n, \delta} \wedge t_{i+1}}^n \right](\omega), \quad i = i_k, \dots, 2^k - 1. \quad (\text{A.6})$$

For any $r \in [t_{i_k+1}, T]$, let j_r be the largest integer such that $t_{j_r} \leq r$. Since $\nu^{n, \delta}$ is an \mathbf{F} -optional time, one can deduce that $\{\nu_k^{n, \delta} \leq r\} = \bigcup_{i=i_k+1}^{j_r} \{\nu_k^{n, \delta} = t_i\} = \bigcup_{i=i_k+1}^{j_r} \{t_{i-1} \leq \nu^{n, \delta} < t_i\} = \{\nu^{n, \delta} < t_{j_r}\} \in \mathcal{F}_{t_{j_r}} \subset \mathcal{F}_r$. So $\nu_k^{n, \delta}$ is a $\mathcal{T}_t(k)$ -stopping time.

(i) Let $i = i_k$ first. We simply denote t_{i_k+1} by s . Since $V_t^n(\omega) > L_t(\omega)$, applying (A.4) with $i = i_k$ yields that

$$V_t^n(\omega) \leq \underline{\mathcal{E}}_t \left[V_s^n + \int_t^s g_r dr \right](\omega). \quad (\text{A.7})$$

As $\nu_k^{n, \delta} \geq t_{i_k+1} = s > t_{i_k} = t$, the first equality in (7.4) shows that

$$\begin{aligned} (\Upsilon_{\nu_k^{n, \delta} \wedge s}^n)^{t, \omega}(\tilde{\omega}) &= \Upsilon^n(\nu_k^{n, \delta}(\omega \otimes_t \tilde{\omega}) \wedge s, \omega \otimes_t \tilde{\omega}) = \Upsilon^n(s, \omega \otimes_t \tilde{\omega}) = V^n(s, \omega \otimes_t \tilde{\omega}) + \int_t^s g_r(\omega \otimes_t \tilde{\omega}) dr + \int_0^t g_r(\omega) dr \\ &= \left(V_s^n + \int_t^s g_r dr \right)^{t, \omega}(\tilde{\omega}) + \int_0^t g_r(\omega) dr, \quad \forall \tilde{\omega} \in \Omega^t. \end{aligned}$$

Taking expectation $\mathbb{E}_{\mathbb{P}}[\cdot]$ and then taking infimum over $\mathbb{P} \in \mathcal{P}(t, \omega)$, we see from (A.7) that

$$\underline{\mathcal{E}}_t \left[\Upsilon_{\nu_k^{n, \delta} \wedge s}^n \right](\omega) = \underline{\mathcal{E}}_t \left[V_s^n + \int_t^s g_r dr \right](\omega) + \int_0^t g_r(\omega) dr \geq \Upsilon_t^n(\omega) = \Upsilon_{\nu_k^{n, \delta} \wedge t}^n(\omega), \quad \text{proving (A.6) for } i = i_k.$$

(ii) Next, let $i = i_k + 1, \dots, 2^k - 1$. Given $\omega \in \{\nu_k^{n, \delta} \leq t_i\}$, applying Proposition 1.1 (3) with $(t, s, \tau) = (0, t_i, \nu_k^{n, \delta})$ shows that $\nu_k^{n, \delta}(\omega \otimes_{t_i} \Omega^{t_i}) \equiv \nu_k^{n, \delta}(\omega) := \hat{t}$. As $\Upsilon_{\hat{t}}^n \in \mathcal{F}_{\hat{t}} \subset \mathcal{F}_{t_i}$, using (1.8) with $(t, s, \eta) = (0, t_i, \Upsilon_{\hat{t}}^n)$ yields that for any $\tilde{\omega} \in \Omega^{t_i}$

$(\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n)^{t_i, \omega}(\tilde{\omega}) = \Upsilon^n(\nu_k^{n,\delta}(\omega \otimes_{t_i} \tilde{\omega}) \wedge t_{i+1}, \omega \otimes_{t_i} \tilde{\omega}) = \Upsilon^n(\hat{t} \wedge t_{i+1}, \omega \otimes_{t_i} \tilde{\omega}) = \Upsilon^n(\hat{t}, \omega \otimes_{t_i} \tilde{\omega}) = \Upsilon^n(\hat{t}, \omega)$. It follows that

$$\underline{\mathcal{E}}_{t_i}[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n](\omega) = \inf_{\mathbb{P} \in \mathcal{P}(t_i, \omega)} \mathbb{E}_{\mathbb{P}}[(\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n)^{t_i, \omega}] = \inf_{\mathbb{P} \in \mathcal{P}(t_i, \omega)} \mathbb{E}_{\mathbb{P}}[\Upsilon^n(\hat{t}, \omega)] = \Upsilon^n(\hat{t}, \omega) = \Upsilon^n(\nu_k^{n,\delta}(\omega) \wedge t_i, \omega). \quad (\text{A.8})$$

Then we let $\omega \in \{\nu_k^{n,\delta} > t_i\}$. Proposition 1.1 (3) shows that

$$\omega \otimes_{t_i} \Omega^{t_i} \subset \{\nu_k^{n,\delta} > t_i\} = \{\nu_k^{n,\delta} \geq t_{i+1}\}, \quad (\text{A.9})$$

and one can deduce that $\nu^{n,\delta}(\omega) \geq t_i \geq t_{i_k+1} > t_{i_k} = t$. By the definition of $\nu^{n,\delta}$, one has $t_i \leq \nu^{n,\delta}(\omega) = \tau_{(t', \omega)}^{n,\delta}(\Pi_{t'}^0(\omega)) \wedge \zeta(\omega) \leq \tau_{(t', \omega)}^{n,\delta}(\Pi_{t'}^0(\omega))$ and it follows that

$$V^n(t_i, \omega) \geq L(t_i, \omega) + \delta. \quad (\text{A.10}^*)$$

This together with (A.4) shows that $V_{t_i}^n(\omega) \leq \underline{\mathcal{E}}_{t_i}[V_{t_{i+1}}^n + \int_{t_i}^{t_{i+1}} g_r dr](\omega)$. Adding $\int_0^{t_i} g_r(\omega) dr$ to both sides, one can deduce from (A.9) that

$$\underline{\mathcal{E}}_{t_i}[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n](\omega) = \underline{\mathcal{E}}_{t_i}[\Upsilon_{t_{i+1}}^n](\omega) = \underline{\mathcal{E}}_{t_i}[V_{t_{i+1}}^n + \int_{t_i}^{t_{i+1}} g_r dr](\omega) + \int_0^{t_i} g_r(\omega) dr \geq \Upsilon_{t_i}^n(\omega) = \Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n(\omega),$$

which together with (A.8) proves (A.6) for $i = i_k + 1, \dots, 2^k - 1$.

3c) As a consequence of (A.6), one then has

$$\underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n](\omega) \leq \underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n](\omega), \quad i = i_k + 1, \dots, 2^k - 1. \quad (\text{A.11})$$

Let $i = i_k + 1, \dots, 2^k - 1$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$. As $\xi_i := \Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n$ is \mathcal{F}_T -measurable by Remark 3.2, Proposition 1.1 (1) shows that $\eta_i := \xi_i^{t_i, \omega}$ is \mathcal{F}_T^t -measurable. Since (3.3) and the first equality in (7.4) show that for any $\tilde{\omega} \in \Omega^t$

$$|\eta_i(\tilde{\omega})| \leq \Psi(\nu_k^{n,\delta}(\omega \otimes_t \tilde{\omega}) \wedge t_{i+1}^k, \omega \otimes_t \tilde{\omega}) + \int_0^T |g_r(\omega \otimes_t \tilde{\omega})| dr \leq \sup_{r \in [t, T]} \Psi_r(\omega \otimes_t \tilde{\omega}) + \int_0^t |g_r(\omega)| dr + \int_t^T |g_r^{t, \omega}(\tilde{\omega})| dr, \quad (\text{A.12})$$

an analogy to (7.13) and (1.9) imply that for all $\tilde{\omega} \in \Omega^t$ except on a \mathbb{P} -null set \mathcal{N}_i ,

$$\mathbb{E}_{\mathbb{P}^{t_i, \tilde{\omega}}}[\eta_i^{t_i, \tilde{\omega}}] = \mathbb{E}_{\mathbb{P}}[\eta_i | \mathcal{F}_{t_i}^t](\tilde{\omega}) \in \mathbb{R}. \quad (\text{A.13})$$

By (P2), there exists an extension $(\Omega^t, \mathcal{F}', \mathbb{P}')$ of $(\Omega^t, \mathcal{F}_T^t, \mathbb{P})$ and $\Omega' \in \mathcal{F}'$ with $\mathbb{P}'(\Omega') = 1$ such that $\mathbb{P}^{t_i, \tilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \tilde{\omega})$ for any $\tilde{\omega} \in \Omega'$. Given $\tilde{\omega} \in \Omega' \cap \mathcal{N}_i^c$, since

$$\eta_i^{t_i, \tilde{\omega}}(\hat{\omega}) = \eta_i(\tilde{\omega} \otimes_{t_i} \hat{\omega}) = \xi_i^{t_i, \omega}(\tilde{\omega} \otimes_{t_i} \hat{\omega}) = \xi_i(\omega \otimes_t (\tilde{\omega} \otimes_{t_i} \hat{\omega})) = \xi_i((\omega \otimes_t \tilde{\omega}) \otimes_{t_i} \hat{\omega}) = \xi_i^{t_i, \omega \otimes_t \tilde{\omega}}(\hat{\omega}), \quad \forall \hat{\omega} \in \Omega^s,$$

we can deduce from (A.6) and (A.13) that

$$\begin{aligned} (\Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n)^{t_i, \omega}(\tilde{\omega}) &= (\Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n)(\omega \otimes_t \tilde{\omega}) \leq \underline{\mathcal{E}}_{t_i}[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n](\omega \otimes_t \tilde{\omega}) = \underline{\mathcal{E}}_{t_i}[\xi_i](\omega \otimes_t \tilde{\omega}) = \inf_{\mathbb{P} \in \mathcal{P}(t_i, \omega \otimes_t \tilde{\omega})} \mathbb{E}_{\mathbb{P}}[\xi_i^{t_i, \omega \otimes_t \tilde{\omega}}] \\ &\leq \mathbb{E}_{\mathbb{P}^{t_i, \tilde{\omega}}}[\xi_i^{t_i, \omega \otimes_t \tilde{\omega}}] = \mathbb{E}_{\mathbb{P}^{t_i, \tilde{\omega}}}[\eta_i^{t_i, \tilde{\omega}}] = \mathbb{E}_{\mathbb{P}}[\eta_i | \mathcal{F}_{t_i}^t](\tilde{\omega}) = \mathbb{E}_{\mathbb{P}}[\xi_i^{t_i, \omega} | \mathcal{F}_{t_i}^t](\tilde{\omega}), \end{aligned}$$

which shows that $\Omega' \cap \mathcal{N}_i^c \subset \mathcal{A}' := \{(\Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n)^{t_i, \omega} \leq \mathbb{E}_{\mathbb{P}}[\xi_i^{t_i, \omega} | \mathcal{F}_{t_i}^t]\} \in \mathcal{F}_T^t$. It follows that $\mathbb{P}(\mathcal{A}') = \mathbb{P}'(\mathcal{A}') \geq \mathbb{P}'(\Omega' \cap \mathcal{N}_i^c) = 1$.

Hence, $(\Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n)^{t_i, \omega} \leq \mathbb{E}_{\mathbb{P}}[\xi_i^{t_i, \omega} | \mathcal{F}_{t_i}^t]$, \mathbb{P} -a.s. Taking the expectation $\mathbb{E}_{\mathbb{P}}[\cdot]$ yields that $\mathbb{E}_{\mathbb{P}}[(\Upsilon_{\nu_k^{n,\delta} \wedge t_i}^n)^{t_i, \omega}] \leq \mathbb{E}_{\mathbb{P}}[\xi_i^{t_i, \omega}] = \mathbb{E}_{\mathbb{P}}[(\Upsilon_{\nu_k^{n,\delta} \wedge t_{i+1}}^n)^{t_i, \omega}]$. Then taking infimum over $\mathbb{P} \in \mathcal{P}(t, \omega)$, we obtain (A.11).

3d) Finally, we will use (A.11) as well as the continuity of process \bar{V} to reach (A.1) for the case $t < T$ with $\bar{V}_t(\omega) > L_t(\omega)$.

Taking $i = i_k$ in (A.6) shows that $\Upsilon_t^n(\omega) = \Upsilon_{\nu_k^{n,\delta} \wedge t}^n(\omega) \leq \underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i_k+1}}^n](\omega)$, which together with (A.11) and (3.1) yields that

$$\Upsilon_t^n(\omega) \leq \underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i_k+1}}^n](\omega) \leq \underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta} \wedge t_{i_k+2}}^n](\omega) \leq \dots \leq \underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta} \wedge t_{2^k}}^n](\omega) = \underline{\mathcal{E}}_t[\Upsilon_{\nu_k^{n,\delta}}^n](\omega) \leq \underline{\mathcal{E}}_t[\bar{\Upsilon}_{\nu_k^{n,\delta}}](\omega). \quad (\text{A.14})$$

Since $\lim_{k \rightarrow \infty} \downarrow \nu_k^{n,\delta} = \nu^{n,\delta}$, the continuity of \bar{V} by Proposition 3.4 implies that $\lim_{k \rightarrow \infty} \bar{\Upsilon}_{\nu_k^{n,\delta}} = \bar{\Upsilon}_{\nu^{n,\delta}}$. Also, an analogy to (A.12) that for any $\tilde{\omega} \in \Omega^t$

$$\left| (\bar{\Upsilon}_{\nu_k^{n,\delta}})^{t,\omega}(\tilde{\omega}) \right| \leq \Psi_*^{t,\omega}(\tilde{\omega}) + \int_0^t |g_r(\omega)| dr + \int_t^T |g_r^{t,\omega}(\tilde{\omega})| dr. \quad (\text{A.15})$$

Then for any $\mathbb{P} \in \mathcal{P}(t, \omega)$, the dominated convergence theorem and an analogy to (7.13) imply that $\lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu_k^{n,\delta}})^{t,\omega} \right] = \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right]$. Taking infimum over $\mathbb{P} \in \mathcal{P}(t, \omega)$ and letting $k \rightarrow \infty$ in (A.14), we obtain

$$\Upsilon_t^n(\omega) \leq \overline{\lim}_{k \rightarrow \infty} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu_k^{n,\delta}})^{t,\omega} \right] \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \overline{\lim}_{k \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu_k^{n,\delta}})^{t,\omega} \right] = \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right].$$

As $\|\omega\|_{0,t} \leq \|\omega\|_{0,T} < \alpha$, we further see from (3.7) that

$$\bar{\Upsilon}_t(\omega) \leq \Upsilon_t^n(\omega) + \rho_\alpha(2^{-n}) + 2^{-n}(|g_t(\omega)| + \rho_\alpha(T-t)) \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right] + \rho_\alpha(2^{-n}) + 2^{-n}(|g_t(\omega)| + \rho_\alpha(T-t)). \quad (\text{A.16})$$

The path regularity of V^n in Proposition 3.4 implies that

$$\lim_{\delta \rightarrow 0} \uparrow \lim_{n \rightarrow \infty} \uparrow \tau_{(t,\omega)}^{n,\delta}(\tilde{\omega}) = \tau_{(t,\omega)}^*(\tilde{\omega}), \quad \forall \tilde{\omega} \in \Omega^t. \quad (\text{A.17}^*)$$

The continuity of \bar{V} thus shows that $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \bar{\Upsilon}_{\nu^{n,\delta}} = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \bar{\Upsilon}_{(\tau_{(t',\omega)}^{n,\delta}(\Pi_{t'}^0) \wedge \zeta) \vee t} = \bar{\Upsilon}_{(\tau_{(t',\omega)}^*(\Pi_{t'}^0) \wedge \zeta) \vee t}$. Also, letting $k \rightarrow \infty$ in (A.15) yields that $|(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega}| \leq \Psi_*^{t,\omega} + \int_0^t |g_r(\omega)| dr + \int_t^T |g_r^{t,\omega}| dr$. Then for any $\mathbb{P} \in \mathcal{P}(t, \omega)$, applying the dominated convergence theorem and an analogy to (7.13) again, we obtain that $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right] = \mathbb{E}_{\mathbb{P}} \left[\left(\bar{\Upsilon}_{(\tau_{(t',\omega)}^*(\Pi_{t'}^0) \wedge \zeta) \vee t} \right)^{t,\omega} \right]$. Eventually, letting $n \rightarrow \infty$ and $\delta \rightarrow 0$ in (A.16) yields that

$$\begin{aligned} \bar{\Upsilon}_t(\omega) &\leq \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right] \leq \overline{\lim}_{\delta \rightarrow 0} \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \overline{\lim}_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right] \leq \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[(\bar{\Upsilon}_{\nu^{n,\delta}})^{t,\omega} \right] \\ &= \inf_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{E}_{\mathbb{P}} \left[\left(\bar{\Upsilon}_{(\tau_{(t',\omega)}^*(\Pi_{t'}^0) \wedge \zeta) \vee t} \right)^{t,\omega} \right]. \end{aligned} \quad \square$$

A.2 Proofs of Starred Statements in Section 7

Proof of (7.11): When $n = \infty$, applying (7.10) with $A = \{\tau \wedge \gamma \geq s\} \in \mathcal{F}_s^t$ and $\tau = \tau \vee s \in \mathcal{T}_s^t$ shows that

$$\sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}_\lambda} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_j} R^{t,\omega}(\tau, \wp_j^n) \right] = \sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}_\lambda} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_j} R^{t,\omega}(\tau \vee s, \wp_j^n) \right] \leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_0^s} \left(V_s^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + \varepsilon.$$

On the other hand, if $n < \infty$, let i_s be the smallest integer such that $i_s 2^{-n} T \geq s$. Clearly, $\tau \vee (i_s 2^{-n} T) \in \mathcal{T}_s^t(n)$. Since $\{\tau \wedge \gamma \geq s\} \subset \{\tau \geq s\} = \{\tau \geq i_s 2^{-n} T\}$, applying (7.10) again with $A = \{\tau \wedge \gamma \geq s\}$ and $\tau = \tau \vee (i_s 2^{-n} T)$ yields that

$$\sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}_\lambda} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_j} R^{t,\omega}(\tau, \wp_j^n) \right] = \sum_{j=1}^{\lambda} \mathbb{E}_{\mathbb{P}_\lambda} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_j} R^{t,\omega}(\tau \vee (i_s 2^{-n} T), \wp_j^n) \right] \leq \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau \wedge \gamma \geq s\} \cap \mathcal{A}_0^s} \left((V_s^n)^{t,\omega} + \int_t^s g_r^{t,\omega} dr \right) \right] + \varepsilon.$$

Proof of (7.12): We set $A_0^s := \{\gamma < s\} \cup (\{\gamma \geq s\} \cap \mathcal{A}_0) \in \mathcal{F}_s^t$ and $A_j^s := \{\gamma \geq s\} \cap \mathcal{A}_j \in \mathcal{F}_s^t$. Given $r \in [t, T]$,

$$\{\hat{\gamma}_\lambda \leq r\} = (A_0^s \cap \{\gamma \leq r\}) \cup \left(\bigcup_{j=1}^{\lambda} (A_j^s \cap \{\wp_j^n \leq r\}) \right). \quad (\text{A.18})$$

If $r < s$, since $\{\gamma \leq r\} \subset \{\gamma < s\}$ and since each $\wp_j^n \in \mathcal{T}_s^t$, one has $\{\hat{\gamma}_\lambda \leq r\} = \{\gamma < s\} \cap \{\gamma \leq r\} = \{\gamma \leq r\} \in \mathcal{F}_r^t$. Otherwise, if $r \geq s$, as $A_j^s \in \mathcal{F}_s^t \subset \mathcal{F}_r^t$ for $j = 0, 1, \dots, \lambda$, (A.18) also implies that $\{\hat{\gamma}_\lambda \leq r\} \in \mathcal{F}_r^t$. Hence, $\hat{\gamma}_\lambda \in \mathcal{T}^t$. \square

Proof of (7.23): Since $\zeta_{\tilde{\omega}} = \lim_{k \rightarrow \infty} \downarrow \zeta_{\tilde{\omega}}^k$, we see that $\{\zeta_{\tilde{\omega}} < \gamma_{\tilde{\omega}}\} \subset A_{\tilde{\omega}} := \bigcup_{k \in \mathbb{N}} \{\zeta_{\tilde{\omega}}^k \leq \gamma_{\tilde{\omega}}\} \subset \{\zeta_{\tilde{\omega}} \leq \gamma_{\tilde{\omega}}\}$ and thus $\{\zeta_{\tilde{\omega}} \leq \gamma_{\tilde{\omega}}\} \setminus A_{\tilde{\omega}} \subset \{\zeta_{\tilde{\omega}} = \gamma_{\tilde{\omega}}\}$. Then the continuity of process L implies that

$$\begin{aligned} R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}, \gamma_{\tilde{\omega}}) &= \int_s^{\zeta_{\tilde{\omega}} \wedge \gamma_{\tilde{\omega}}} g_r^{s, \omega \otimes_t \tilde{\omega}} dr + \mathbf{1}_{\{\zeta_{\tilde{\omega}} \leq \gamma_{\tilde{\omega}}\}} L_{\zeta_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} + \mathbf{1}_{\{\gamma_{\tilde{\omega}} < \zeta_{\tilde{\omega}}\}} U_{\gamma_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} \\ &= \int_s^{\zeta_{\tilde{\omega}} \wedge \gamma_{\tilde{\omega}}} g_r^{s, \omega \otimes_t \tilde{\omega}} dr + \mathbf{1}_{A_{\tilde{\omega}}} L_{\zeta_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} + \mathbf{1}_{\{\zeta_{\tilde{\omega}} \leq \gamma_{\tilde{\omega}}\} \setminus A_{\tilde{\omega}}} L_{\zeta_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} + \mathbf{1}_{\{\gamma_{\tilde{\omega}} < \zeta_{\tilde{\omega}}\}} U_{\gamma_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} \leq \int_s^{\zeta_{\tilde{\omega}} \wedge \gamma_{\tilde{\omega}}} g_r^{s, \omega \otimes_t \tilde{\omega}} dr + \mathbf{1}_{A_{\tilde{\omega}}} L_{\zeta_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} + \mathbf{1}_{A_{\tilde{\omega}}^c} U_{\gamma_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} \\ &= \lim_{k \rightarrow \infty} \left(\int_s^{\zeta_{\tilde{\omega}}^k \wedge \gamma_{\tilde{\omega}}} g_r^{s, \omega \otimes_t \tilde{\omega}} dr + \mathbf{1}_{\{\zeta_{\tilde{\omega}}^k \leq \gamma_{\tilde{\omega}}\}} L_{\zeta_{\tilde{\omega}}^k}^{s, \omega \otimes_t \tilde{\omega}} + \mathbf{1}_{\{\gamma_{\tilde{\omega}} < \zeta_{\tilde{\omega}}^k\}} U_{\gamma_{\tilde{\omega}}}^{s, \omega \otimes_t \tilde{\omega}} \right) = \lim_{k \rightarrow \infty} R^{s, \omega \otimes_t \tilde{\omega}}(\zeta_{\tilde{\omega}}^k, \gamma_{\tilde{\omega}}). \end{aligned}$$

Proof of (7.24): For any $\tau_1, \tau_2 \in \mathcal{T}_s^t$, letting $A := \{\mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_1, \hat{\gamma}') | \mathcal{F}_s^t] \geq \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_2, \hat{\gamma}') | \mathcal{F}_s^t]\} \in \mathcal{F}_s^t$ and $\bar{\tau} := \mathbf{1}_{A\tau_1} + \mathbf{1}_{A^c}\tau_2 \in \mathcal{T}_s^t$, we can deduce that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\bar{\tau}, \hat{\gamma}') | \mathcal{F}_s^t] &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A R^{t, \omega}(\tau_1, \hat{\gamma}') + \mathbf{1}_{A^c} R^{t, \omega}(\tau_2, \hat{\gamma}') | \mathcal{F}_s^t] = \mathbf{1}_A \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_1, \hat{\gamma}') | \mathcal{F}_s^t] + \mathbf{1}_{A^c} \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_2, \hat{\gamma}') | \mathcal{F}_s^t] \\ &= \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_1, \hat{\gamma}') | \mathcal{F}_s^t] \vee \mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau_2, \hat{\gamma}') | \mathcal{F}_s^t]. \end{aligned}$$

So the family $\{\mathbb{E}_{\mathbb{P}}[R^{t, \omega}(\tau, \hat{\gamma}') | \mathcal{F}_s^t]\}_{\tau \in \mathcal{T}_s^t}$ is directed upwards. Appealing to the basic properties of the essential infimum (e.g., [48, Proposition VI-1-1]), we can find a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in \mathcal{T}_s^t such that (7.24) holds. \square

Proof of (7.25): For any $r \in [t, s)$, since $\tau_n \in \mathcal{T}_s^t$ and since $\{\tau \leq r\} \subset \{\tau < s\} \subset \{\tau \wedge \hat{\gamma} < s\}$, one can deduce that $\{\bar{\tau}_n \leq r\} = \{\tau \wedge \hat{\gamma} < s\} \cap \{\tau \leq r\} = \{\tau \leq r\} \in \mathcal{F}_r^t$. On the other hand, for any $r \in [s, T]$, $\{\bar{\tau}_n \leq r\} = (\{\tau \wedge \hat{\gamma} < s\} \cap \{\tau \leq r\}) \cup (\{\tau \wedge \hat{\gamma} \geq s\} \cap \{\tau_n \leq r\}) \in \mathcal{F}_r^t$. Hence, $\bar{\tau}_n \in \mathcal{T}^t$.

Proof of (7.49): Given $r \in [s, T]$, as $A_r := \{\varphi \leq r\} \in \mathcal{F}_r^s$, (5.4) shows that

$$\{\varphi_j \leq r\} = \{\hat{\omega} \in \Omega^s : \varphi(\mathcal{X}^j(\hat{\omega})) \leq r\} = \{\hat{\omega} \in \Omega^s : \mathcal{X}^j(\hat{\omega}) \in A_r\} = (\mathcal{X}^j)^{-1}(A_r) \in \bar{\mathcal{F}}_r^s. \quad (\text{A.19})$$

Also, Lemma A.3 in the ArXiv version of [5] implies that $\{\nu_j \leq r\} = \{\tilde{\omega} \in \Omega^t : \Pi_s^t(\tilde{\omega}) \in \{\varphi_j \leq r\}\} = (\Pi_s^t)^{-1}(\{\varphi_j \leq r\}) \in \bar{\mathcal{F}}_r^t$, then one can deduce from (7.45) that $\{\hat{\gamma}_j \leq r\} = \{\tilde{\omega} \in \Omega^t : \widehat{\mathcal{W}}(\tilde{\omega}) \in \{\nu_j \leq r\}\} = \widehat{\mathcal{W}}^{-1}(\{\nu_j \leq r\}) \in \widehat{\mathfrak{F}}_r$. Hence, $\varphi_j \in \bar{\mathcal{T}}^s$, $\nu_j \in \bar{\mathcal{T}}_s^t$ while $\hat{\gamma}_j$ is a $\widehat{\mathfrak{F}}$ -stopping time that takes values in $[s, T]$. \square

Proof of (7.58): When $n < \infty$, as induced by $\tau \in \mathcal{T}_s^t(n)$, $\zeta_{\tilde{\omega}}$ takes values in $\{t_i^n\}_{i=i_s}^{2^n}$, where i_s be the smallest integer such that $i_s 2^{-n} T \geq s$. Similar to (7.50), there exists $\zeta_{\tilde{\omega}}' \in \mathcal{T}^s(n)$ such that $\zeta_{\tilde{\omega}}' = \zeta_{\tilde{\omega}}$, \mathbf{p}_j -a.s. So we have

$$\mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}, \varphi)] = \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}', \varphi)] = \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}', \varphi)] \leq \sup_{\varsigma \in \mathcal{T}^s(n)} \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\varsigma, \varphi)].$$

Suppose $n = \infty$ now. Let $k \in \mathbb{N}$ and set $s_i^k := s \vee (i 2^{-k} T)$, $i = 0, \dots, 2^k$. With $s_{-1}^k := -1$, $\zeta_{\tilde{\omega}}^k := \sum_{i=0}^{2^k} \mathbf{1}_{\{s_{i-1}^k < \zeta_{\tilde{\omega}} \leq s_i^k\}} s_i^k$ defines a $\widehat{\mathfrak{F}}^j$ -stopping time. By similar arguments to those that lead to (7.50), one can construct a \mathcal{T}^s -stopping time $\zeta_{\tilde{\omega}}^k$ valued in $\{s_i^k\}_{i=0}^{2^k}$ such that $\zeta_{\tilde{\omega}}^k = \zeta_{\tilde{\omega}}$, \mathbf{p}^j -a.s. Since $\zeta_{\tilde{\omega}} = \lim_{k \rightarrow \infty} \downarrow \zeta_{\tilde{\omega}}^k$, an analogy to (7.23) shows that

$$R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}, \varphi) \leq \lim_{k \rightarrow \infty} R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}^k, \varphi). \quad (\text{A.20})$$

By (2.5), $|R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}^k, \varphi)| \leq \int_s^T |g_r^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}| dr + \Psi_*^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}$, $\forall k \in \mathbb{N}$. Since $\mathbf{p}_j \in \widehat{\mathfrak{P}}_s$ by (7.43), (2.6) shows that

$$\mathbb{E}_{\mathbf{p}^j} \left[\int_s^T |g_r^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}| dr + \Psi_*^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})} \right] = \mathbb{E}_{\mathbf{p}^j} \left[\int_s^T |g_r^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}| dr + \Psi_*^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})} \right] < \infty.$$

Taking expectation $\mathbb{E}_{\mathbf{p}^j}[\cdot]$ in (A.20), one can deduce from the dominated convergence theorem that

$$\begin{aligned} \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}, \varphi)] &\leq \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}^k, \varphi)] = \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}^k, \varphi)] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\zeta_{\tilde{\omega}}^k, \varphi)] \leq \sup_{\varsigma \in \mathcal{T}^s} \mathbb{E}_{\mathbf{p}^j} [R^{s, \omega \otimes_t \mathcal{X}(\tilde{\omega})}(\varsigma, \varphi)]. \end{aligned} \quad \square$$

Proof of (A.10): If $t_i < \tau_{(t',\omega)}^{n,\delta}(\Pi_{t'}^0(\omega))$, the definition of $\tau_{(t',\omega)}^{n,\delta}$ shows that $(V^n - L)(t_i, \omega) = ((V^n)^{t',\omega} - L^{t',\omega})(t_i, \Pi_{t'}^0(\omega)) \geq \delta \geq 0$. On the other hand, if $t_i = \tau_{(t',\omega)}^{n,\delta}(\Pi_{t'}^0(\omega))$ the left-upper-semicontinuity of $(V^n)^{t',\omega} - L^{t',\omega}$ implies that

$$(V^n - L)(t_i, \omega) = ((V^n)^{t',\omega} - L^{t',\omega})(t_i, \Pi_{t'}^0(\omega)) \geq \overline{\lim}_{s \nearrow t_i} ((V^n)^{t',\omega} - L^{t',\omega})(s, \Pi_{t'}^0(\omega)) \geq \delta. \quad \square$$

Proof of (A.17): Fix $\tilde{\omega} \in \Omega^t$ and set $\tilde{\alpha} := 1 + \|\omega \otimes_t \tilde{\omega}\|_{0,T}$.

We Let $\delta > 0$, $n \in \mathbb{N}$ and simply denote $t_{n,\delta} := \tau_{(t,\omega)}^{n,\delta}(\tilde{\omega})$, $t_* := \tau_{(t,\omega)}^*(\tilde{\omega})$. Let us first show that

$$(V^n)^{t,\omega}(t_{n,\delta}, \tilde{\omega}) \leq L^{t,\omega}(t_{n,\delta}, \tilde{\omega}) + \delta. \quad (\text{A.21})$$

If $t_{n,\delta} = T$, (3.2) shows that

$$(V^n)^{t,\omega}(t_{n,\delta}, \tilde{\omega}) = (V^n)^{t,\omega}(T, \tilde{\omega}) = L^{t,\omega}(T, \tilde{\omega}) = L^{t,\omega}(t_{n,\delta}, \tilde{\omega}). \quad (\text{A.22})$$

On the other hand, if $t_{n,\delta} < T$, let $\{t_i = t_i(t, \omega, \tilde{\omega}, n, \delta)\}_{i \in \mathbb{N}}$ be a sequence in $[t_{n,\delta}, T]$ such that $\lim_{i \rightarrow \infty} \downarrow t_i = t_{n,\delta}$ and that $(V^n)^{t,\omega}(t_i, \tilde{\omega}) < L^{t,\omega}(t_i, \tilde{\omega}) + \delta$, $\forall i \in \mathbb{N}$ by the definition of $t_{n,\delta} = \tau_{(t,\omega)}^{n,\delta}(\tilde{\omega})$. The right-lower-semicontinuity of path $V^n(\omega \otimes_t \tilde{\omega})$ by Proposition 3.4 and the continuity of path $L(\omega \otimes_t \tilde{\omega})$ then imply that

$$(V^n)^{t,\omega}(t_{n,\delta}, \tilde{\omega}) = V^n(t_{n,\delta}, \omega \otimes_t \tilde{\omega}) \leq \underline{\lim}_{s \searrow t_{n,\delta}} V^n(s, \omega \otimes_t \tilde{\omega}) \leq \underline{\lim}_{i \rightarrow \infty} V^n(t_i, \omega \otimes_t \tilde{\omega}) \leq L(t_{n,\delta}, \omega \otimes_t \tilde{\omega}) + \delta = L^{t,\omega}(t_{n,\delta}, \tilde{\omega}) + \delta,$$

which together with (A.22) proves (A.21).

As $\|\omega \otimes_t \tilde{\omega}\|_{0,t_{n,\delta}} \leq \|\omega \otimes_t \tilde{\omega}\|_{0,T} < \tilde{\alpha}$, we see from (A.21) and (3.7) that

$$\bar{V}^{t,\omega}(t_{n,\delta}, \tilde{\omega}) - L^{t,\omega}(t_{n,\delta}, \tilde{\omega}) \leq \bar{V}(t_{n,\delta}, \omega \otimes_t \tilde{\omega}) - V^n(t_{n,\delta}, \omega \otimes_t \tilde{\omega}) + \delta \leq \rho_{\tilde{\alpha}}(2^{-n}) + 2^{-n} \left(\sup_{r \in [0,T]} |g_r(\omega \otimes_t \tilde{\omega})| + \rho_{\tilde{\alpha}}(T) \right) + \delta. \quad (\text{A.23})$$

For any $s \in [t, T]$, since $\mathcal{T}^s(n) \subset \mathcal{T}^s(n+1) \subset \mathcal{T}^s$, an analogy to (3.1) shows that $V_s^n(\omega \otimes_t \tilde{\omega}) \leq V_s^{n+1}(\omega \otimes_t \tilde{\omega}) \leq \bar{V}_s(\omega \otimes_t \tilde{\omega})$. It follows that $\hat{t}_\delta := \lim_{n \rightarrow \infty} \uparrow t_{n,\delta} \leq t_*$. As $n \rightarrow \infty$ in (A.23), the continuity of the path $\bar{V}^{t,\omega}(\tilde{\omega}) - L^{t,\omega}(\tilde{\omega})$ by Proposition 3.4 yields that $\bar{V}^{t,\omega}(\hat{t}_\delta, \tilde{\omega}) - L^{t,\omega}(\hat{t}_\delta, \tilde{\omega}) \leq \delta$, and thus

$$t_* \geq \lim_{n \rightarrow \infty} \uparrow t_{n,\delta} = \hat{t}_\delta \geq t_{*,\delta} := \inf \{s \in [t, T] : \bar{V}_s^{t,\omega}(\tilde{\omega}) \leq L_s^{t,\omega}(\tilde{\omega}) + \delta\}. \quad (\text{A.24})$$

The continuity of the path $\bar{V}^{t,\omega}(\tilde{\omega}) - L^{t,\omega}(\tilde{\omega})$ also implies that $t_* = \lim_{\delta \rightarrow 0} \uparrow t_{*,\delta}$ which together with (A.24) leads to that $\lim_{\delta \rightarrow 0} \uparrow \lim_{n \rightarrow \infty} \uparrow t_{n,\delta} = t_*$, i.e., $\lim_{\delta \rightarrow 0} \uparrow \lim_{n \rightarrow \infty} \uparrow \tau_{(t,\omega)}^{n,\delta}(\tilde{\omega}) = \tau_{(t,\omega)}^*(\tilde{\omega})$. \square

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